A Novel Approach for Spherical Spline Split Quaternion Interpolation on Lorentzian Sphere using Bezier Curve Algorithm

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Abstract: This paper presents the spline split quaternion interpolation on Lorentzian spheres. The split quaternions don't have group structure on the Lorentzian sphere, there for not defined squad (spline split quaternion interpolation in Minkowski space). In this paper, we propose a new method for smoothly interpolation on Lorentzian sphere using orthogonal projection and cubic Bezier curve.
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1. Introduction

Quaternions were discovered by Sir William Rowan Hamilton as an extension to the complex number in 1843. The most important property of quaternions is that every unit quaternion represents a rotation and this plays a special role in the study of rotations in three dimensional spaces. Also quaternions are an efficient way understanding many aspects of physics and kinematics. Many physical laws in classical, relativistic and quantum mechanics can be written nicely using them. Today they are used especially in the area of computer vision, computer graphics, animations, aerospace applications, flight simulators, navigation systems and to solve optimization problems involving the estimation of rigid body transformations. Shoemake (Shoemake 1985) suggested spherical linear interpolation (slerp) as a means for determining the intermediate orientations between two given ones. In computer graphics, slerp is shorth and for spherical linear interpolation, in the context of quaternion interpolation for the purpose of animating 3D rotation. It is a fundamental problem of computer animation and other computer simulations involving the dynamics of rigid bodies to be able to smoothly interpolate between a sequence of positions and orientations. Smooth interpolation of three-dimensional object orientation, starting from n key frame orientations, is used in computer animation to model moving solids, cameras, and lights (Noakes). Spherical spline curves have potential applications in computer graphics, in animation and in robotics and motion planning based on quaternions. In (Ghadami *et al.*), we showed spherical spline interpolation on hyperbolic spheres using split quaternions and metric Lorentz. For this reason set of split quaternion H′ is a non group structure on the Lorentzian sphere. The purpose of this paper offers a new method for smoothly quaternion interpolation on Lorentzian sphere using orthogonal projection and cubic Bezier curve. Since Bezier is one of the imperative polynomial and important tool for interpolation Bezier polynomial has several applications fields of engineering, science and technology such as highway or railway rout designing, networks, computer aided design system, animation, robotics, communications and many other discipline (Abbass and Jamal 2011). Also, in this paper we show propose method on the hyperbolic sphere and Euclidean sphere.

2. Preliminary

In this section, we give some useful definition and propositions about Minkowski space (O'Neill 1983, Kula and Yayli 2007).

Definition 1. The Lorentz-Minkowski space is the metric space $E_1^3 = \left(R^3, \langle , \rangle\right)$, where the metric $\langle , \rangle$ is given

$$\langle u, v\rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$$

The metric $\langle , \rangle$ is called the Lorentzian metric.

A vector $\vec{v} \in E_1^3$ is called

1. spacelike if $\langle v, v\rangle > 0$ or $v = 0$
2. timelike if $\langle v, v\rangle < 0$
3. lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$

We point out that the null vector $v = 0$ is considered of spacelike type although it satisfies $\langle v,v\rangle = 0$. the norm of the vector $\vec{u} \in E_1^3$ is define by $\|\vec{u}\| = \sqrt{|\langle \vec{u},\vec{u}\rangle|}$. The Lorentzian vector product $\vec{u} \wedge \vec{v}$ of $\vec{u}$ and $\vec{v}$ is define as follows:

$$\vec{u} \wedge \vec{v} = \begin{bmatrix} -e_1 & e_2 & e_3 \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

The hyperbolic and Lorentzian unit spheres are $H_0^2 = \{\vec{a} \in E_1^3 : \langle \vec{a},\vec{a}\rangle = -1\}$, $S_1^2 = \{\vec{a} \in E_1^3 : \langle \vec{a},\vec{a}\rangle = 1\}$

Theorem 1. Let $\vec{u}$ and $\vec{v}$ be vectors in the Minkowski 3-space. If $\vec{u}$ and $\vec{v}$ are timelike vectors, the $\vec{u} \wedge \vec{v}$ is a spacelike vector. $\langle \vec{u},\vec{v}\rangle = -\|\vec{u}\|\|\vec{v}\|\cosh\varphi$ and $\|\vec{u} \wedge \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sinh\varphi$ where φ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$. The set of timelike vectors will be denote by τ and it is the following set: $\tau = \{(x,y,z) \in E_1^3 ; x^2 + y^2 - z^2 < 0\}$.

Proposition 1. Two timelike vectors $\vec{u}$ and $\vec{v}$ lie in the same timelike cone if and only if $\langle \vec{u},\vec{v}\rangle < 0$.

3. Split quaternion

Definition 2. The algebra H' of split quaternion is defined as the 4-dimensional vector space over R having a basis $\{1,i,j,k\}$ with the following properties (Kula and Yayli 2007)

$$i^2 = -1, j^2 = k^2 = 1$$

$$ij = -ji = k, kj = -jk = -i, ki = -ik = j.$$

form it is clear that H' is not commutative and 1 is the identity element of H' .It also H' is an associative algebra. For

$$q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k \in H'$$

$$(a_0,a_1,a_2,a_3 \in R)$$

we define the conjugate $\bar{q}$ of q as

$\bar{q} = a_0 \cdot 1 - a_1 \cdot i - a_2 \cdot j - a_3 \cdot k \in H'$. For every $q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k \in H'$ *we have* $q \cdot \bar{q} = (a_0^2 + a_1^2 - a_2^2 - a_3^2)$

we define the norm N_q and the inverse q^{-1} of the quaternion respectively the real number $N_q = a_0^2 + a_1^2 - a_2^2 - a_3^2$ and $q^{-1} = \frac{\bar{q}}{N_q}, N_q \neq 0$.

If $N_q = 1$ then q is called unit split quaternion. The algebra H_1' of split quaternion is called unit split quaternion. If $q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k$ and $p = b_0 \cdot 1 + b_1 \cdot i + b_2 \cdot j + b_3 \cdot k$ be two split quaternion and let $r = qp$, then r is given by

$$r = S_q S_p + g(V_q,V_p) + S_q V_p + S_p V_q + V_q \wedge V_p,$$

where

$$S_q = a_0,\ S_p = b_0,\ g(V_q,V_p) = -a_1b_1 + a_2b_2 + a_3b_3,$$

$$V_q = a_1 \cdot i + a_2 \cdot j + a_3 \cdot k,\ V_p = b_1 \cdot i + b_2 \cdot j + b_3 \cdot k$$

$$V_q \wedge V_p = (a_3b_2 - a_2b_3)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k.$$

If $S_q = 0$ then q is called pure split quaternion. Split quaternion product of two pure split quaternions $q = a_1 \cdot i + a_2 \cdot j + a_3 \cdot k$ and $p = b_1 \cdot i + b_2 \cdot j + b_3 \cdot k$ is

$$qp = \langle V_q,V_p\rangle + V_q \wedge V_p$$

$$= -a_1b_1 + a_2b_2 + a_3b_3 + \begin{bmatrix} -i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

4. Linear Interpolation in Minkowski space

In this section, we compute the interpolation on hyperbolic sphere. We have done this interpolations using metric Lorentz and split quaternion.

Remark 1. The split quaternion, $p,q \in H_1'$, product $p^{-1}q$ can be greatly simplified by use of the fact that, for a unit split quaternion $u = [\cosh\varphi, w\sinh\varphi]$ and $u^t = [\cosh(t\varphi), \sinh(t\varphi)]$. From the definition you can see that $t = 0$ give rotation p, $t = 1$ the rotation q, and $t \in (0,1)$ gives all intermediate rotations (Ghadami *et al., 2012*),.

Proposition 5.

The curve $slerp(p,q,n): H_1' \times H_1' \times [0,1] \to H_1'$ is a great arc on the unit split quaternion hyperbolic sphere between p and q (Ghadami *et al., 2012*),.

Definition 3. The split quaternion used for a starting rotation given by p and ending with rotation q, for $p,q \in H_1'$, $q = p(p^{-1}q)^n$. This can be written

$$slerp(p,q,n) = p(p^{-1}q)^n, n \in [0,1]$$

While from the $4-D$ geometry comes

$$slerp(p,q,n) = \frac{p\sinh((1-n)\varphi) + q\sinh(n\varphi)}{\sinh(\varphi)}$$

$$p,q \in H_1', \quad n \in [0,1]$$

Where $-\langle q_0,q_1\rangle = \cosh\varphi$. Slerp is spherical linear interpolation in Minkowski space (Ghadami *et al., 2012*).

5. Spline Interpolation of Split Quaternion on Hyperbolic Sphere

A Given a sequence on N unit split quaternion $\{q_n\}_{n=0}^{N-1}$, we want to build a spline which interpolated those split quaternion subject to the conditions that the spline pass through the control points and that the derivatives are continuous. The idea is to choose intermadiate split quaternions s_n and s_{n+1}. To allow control of the derivatives at the end points of the spline segements. The points s_n and s_{n+1} are called inner quadrangle points, and have to be chosen carefully so that continuity is guaranteed across segments. More precisely, let

$$squad(q_i, q_{i+1}, s_i, s_{i+1}, n) =$$

$$slerp(slerp(q_i, q_{i+1}, n), slerp(s_i, s_{i+1}, n), 2n(1-n))$$

Be the spline segments. ($squad$ is spherical spline split quaternion interpolation on hyperbolic sphere in Minkowski space) By definition

$$q_i \log(\left(q_{i-1}^{-1}, q_i\right) - 2\log\left(q_i^{-1} s_i\right) =$$

$$q_i \log(\left(q_i^{-1}, q_{i+1}\right) + 2\log\left(q_i^{-1} s_i\right)$$

$$s_i = q_i \exp\left(-\frac{\log\left(q_i, q_{i-1}^{-1}\right) + \log\left(q_i^{-1}, q_{i+1}\right)}{4}\right)$$

Thus $squad$ is continuously differentiable at the control points with s_i defined as above. All in all we have shown that $squad$ is continuous and continuously differentiable across all segments (Ghadami *et al.,*) (Figure 1).

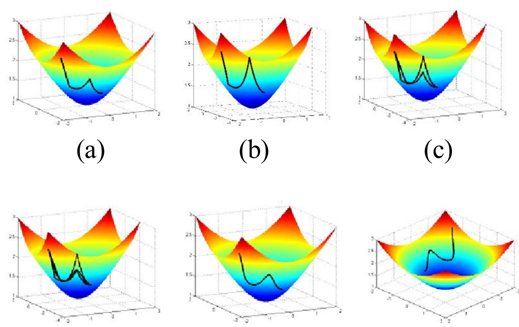


Figure 1. The shapes of interpolation are simulated with MATLAB R2010a, (a) Split quaternion interpolation between the four key frames on hyperbolic sphere, (b) Inner quadrangle interpolation between the four key frames on hyperbolic sphere, (c) Combination of split quaternion and inner quadrangle on the hyperbolic sphere, (d) Smoothing split quaternion with using inner quadrangle, (e) interpolation curve for squad, (f) Inside scope interpolation curve for squad.

6. Interpolation on Lorentzian Sphere using Bezier Curve Algorithm

In this section shows to construct a smooth interpolation on Lorentzian sphere using cubic Bezier curve and orthogonal projection. Since set of split quaternion H' don't have group structure on the Lorentzian sphere. Therefore we can't use spherical spline split quaternion interpolation on hyperbolic sphere method that showed in (Ghadami *et al.*)· In this paper, also we show propose method on the hyperbolic sphere and Euclidean sphere. Bezier curves of any degree can be defined. A degree n Bezier curve has n +1 control points whose blending functions are denoted $B_i^n(t)$, where (Abbass and Jamal 2011)

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i} t^i, \quad i = 0,1,2,\ldots, n.$$

Recall that $\binom{n}{i}$ is called a binomial coefficient, sometimes spoken "n- choose -i", and is equal to $\frac{n!}{i!(n-i)!}$. $B_i^n(t)$ is also referred to as the ith Bernstein polynomial of degree n. The equation of a Bezier curve is thus:

$$R(t) = \sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i} t^i P_i.$$

$P_i = (x_i, y_i)$, $i = 0,1,2,\ldots,n$ is the control points of Bezier curve. In this paper, we use cubic Bezier curve. Cubic Bezier curve is defined as

$$R(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$$
$$0 \le t \le 1$$

The proposed method is explained as under steps: - Select split quaternion on Lorentzian sphere; - Draw Cubic split quaternion interpolation (as a set of three linear interpolations) on Lorentzian sphere; - The split quaternion interpolation curve one to one mapping to plane with orthogonal projection - Smooth the split quaternion interpolation using cubic Bezier algorithm in plane - smoothed curve is taken to Lorentzian sphere with transformation (Figure 1).

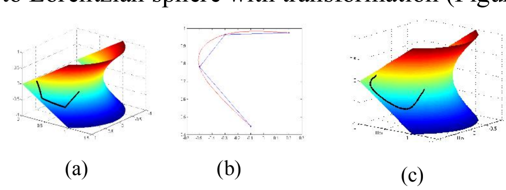
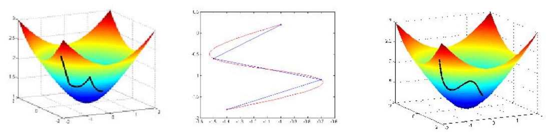
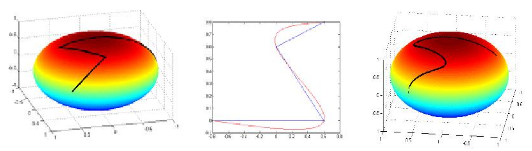


Figure 2. Interpolation on Lorentzian sphere, (a) split quaternion interpolation between the four key frames on Lorentzian sphere, (b) split quaternion interpolation curve mapping to plane with orthogonal projection and smooth the curve using cubic Bezier algorithm, (c) Smoothed curve is taken to Lorentzian sphere with transformation.



(a) (b) (c)

Figure 3. Interpolation on hyperbolic sphere. (a) split quaternion interpolation between the four key frames on hyperbolic sphere, (b) split quaternio interpolation curve mapping to plane with orthogonal projection and smooth the curve using cubic Bezier algorithm, (c) Smoothed curve is taken to hyperbolic sphere with transformation



(a) (b) (c)

Figure 4. Interpolation on Euclidean sphere. (a) Quaternion interpolation between the four key frames on Euclidean sphere, (b) quaternion interpolation curve mapping to plane with orthogonal projection and smooth the curve using cubic Bezier algorithm, (c) Smoothed curve is taken to Euclidean sphere with transformation

4. Discussions

The split quaternions have group structure on the hyperbolic sphere, using these properties, squad (spherical spline split quaternion interpolation in Minkowski space) interpolation is defined on the hyperbolic sphere. But the split quaternions don't have group structure on the Lorentzian sphere, there for not defined squad. This problem, for the Lorentzian sphere solved one to one orthogonal projection to plane and cubic Bezier algorithm. Also, the proposed method by squad interpolation can be used on the hyperbolic sphere and Euclidean sphere. Our results are favorable, but this method can be used with different projection.

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