Sufficient condition of a subclass of analytic functions defined by Hadamard product

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Abstract: In the present article we obtain a sufficient condition for a function belongs to a class of analytic functions defined by convolution. The main result presented here includes a number of known consequences as special cases.
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1. Introduction

Let the class of all functions

$$f(z) = z + a_{n+1}z^{n+1} + \cdots$$

which are analytic in $E = \{z; |z| < 1\}$ be denoted by A_n and let $A_1 = A$.

A function $f(z) \in A_n$ is spiral-like of order β, if

$$\mathrm{Re}\, e^{i\lambda}\frac{zf'(z)}{f(z)} > \beta\cos\lambda, 0 \le \beta < 1,$$

for all $z \in E$ and λ is real with $|\lambda| < \frac{\pi}{2}$. We denoted the class of all such spiral-like functions of order β by $S^*_\lambda(n,\beta)$. For $n = 1$ and $\beta = 0$, this class reduces to the well-known class of spiral-like functions which was introduced by Spacek [4] in 1933.

For any two analytic functions $f(z), g(z) \in A_n$, we define the convolution or Hadamard product by

$$(f * g)(z) = z + a_{n+1}z^{n+1} + \cdots,$$

where $f(z)$ and $g(z)$ are given by

$$f(z) = z + a_{n+1}z^{n+1} + \cdots, \text{ and } g(z) = z + a_{n+1}z^{n+1} + \cdots.$$

Using the concept of convolution, we define a subclass $Q_\lambda(g,n,\beta)$ of analytic functions as follows:

A function $f(z) \in A_n$ belongs to the class $Q_\lambda(g,n,\beta)$, if

$$\mathrm{Re}\, e^{i\lambda}\frac{z(f*g)'(z)}{(f*g)(z)} > \beta\cos\lambda, 0 \le \beta < 1,$$

for all $z \in E$ with $(f*g)(z) \neq 0$ and λ is real with $|\lambda| < \frac{\pi}{2}$. This class gives a transition from the class S^* of starlike functions to the class C of convex functions.

In this paper, we obtain a sufficient condition for a function to be in the class $Q_\lambda(g,n,\beta)$. To prove our main results, we need the following Lemma proved in [2].

Lemma. Let Ω be a set in the complex plane C and suppose that ϕ is a mapping from $C^2 \times E$ to C which satisfies $\phi(ix,y;z) \notin \Omega$ for $z \in E$, and for all real x, y such that $y \le -n(1+x^2)/2$. If $p(z) = 1 + c_n z^n + \cdots$ is analytic in E and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in E$, then $\mathrm{Re}\, p(z) > 0$.

Main results

In this section, we study some sufficient conditions for function belongs $Q_\lambda(g,n,\beta)$.

Theorem 2.1. If $f(z) \in A_n$, satisfies

$$\mathrm{Re}\left(e^{i\lambda}\frac{z(f*g)'(z)}{(f*g)(z)}\right)\left(\frac{\alpha z(f*g)''(z)}{(f*g)'(z)} + 1\right) > \frac{M^2}{4L} + N, (z \in E)$$

where $0 \le \alpha \le 1$, $0 \le \beta < 1$, λ is real with $|\lambda| < \frac{\pi}{2}$

and

$$L = \alpha(1-\beta)\cos\lambda\left[\frac{n}{2} + (1-\beta)\cos^2\lambda\right]$$
$$M = -(1-\beta)^2\sin 2\lambda\cos\lambda$$
$$N = \alpha\cos\lambda(\beta^2\cos^2\lambda - \sin^2\lambda) + \alpha\beta\sin\lambda\sin 2\lambda\,\alpha\cos^2\lambda + \beta(1-\alpha)\cos\lambda - \frac{n\alpha}{2}(1-\beta).$$

Then $f(z) \in Q_\lambda(g,n,\beta)$.

Proof. Set

$$\frac{z(f*g)'(z)}{(f*g)(z)} = q(z) = \cos\lambda[(1-\beta)p(z) + \beta] + i\sin\lambda. \quad (2.1)$$

Then $p(z)$ and $q(z)$ are analytic in E with $p(0) = 1$ and $q(0) = 1$.

Taking logarithmic differentiation of (2.1), we have

$$\frac{z(f*g)''(z)}{(f*g)'(z)} = \frac{zq'(z)+e^{-i\lambda}q^2(z)-q(z)}{q(z)},$$

and hence

$$\left(e^{i\lambda}\frac{z(f*g)'(z)}{(f*g)(z)}\right)\left(\frac{\alpha z(f*g)''(z)}{(f*g)'(z)}+1\right) = Azp'(z)+Bp^2(z)+Cp(z)+D = \phi(p(z),zp'(z);z),$$

with

$$A = \alpha(1-\beta)\cos\lambda,$$
$$B = \alpha e^{-i\lambda}(1-\beta)^2\cos^2\lambda,$$
$$C = (1-\beta)\big(2\alpha\beta e^{-i\lambda}\cos^2\lambda + i\alpha e^{-i\lambda}\sin 2\lambda + (1-\alpha)\cos\lambda\big),$$
$$D = \alpha e^{-i\lambda}(\beta^2\cos^2\lambda - \sin^2\lambda + i\beta\sin 2\lambda) + (1-\alpha)(\beta\cos\lambda + i\sin\lambda).$$

Now

$$\phi(r,s;t) = As + Br^2 + Cr + D.$$

For all real x and y satisfying $y \le -n(1+x^2)/2$, we have

$$\begin{aligned}\phi(ix,y;z) &= Ay + B(ix)^2 + C(ix) + D\\ &= Ay - Bx^2 + iCx + D\\ &\le -\frac{(1+x^2)nA}{2} - Bx^2 + iCx + D\\ &= -\left(\frac{nA}{2}+B\right)x^2 + iCx - \frac{nA}{2} + D\\ &= -\left[\frac{n}{2}\alpha(1-\beta)\cos\lambda + \alpha e^{-i\lambda}(1-\beta)^2\cos^2\lambda\right]x^2\\ &\quad + i\big[(1-\beta)\big(2\alpha\beta e^{-i\lambda}\cos^2\lambda + i\alpha e^{-i\lambda}\sin 2\lambda + (1-\alpha)\cos\lambda\big)\big]x\\ &\quad + \alpha e^{-i\lambda}(\beta^2\cos^2\lambda - \sin^2\lambda + i\beta\sin 2\lambda)\\ &\quad + (1-\alpha)(\beta\cos\lambda + i\sin\lambda).\end{aligned}$$

Now taking real part of both sides, we have

$$\begin{aligned}\mathrm{Re}\,\phi(ix,y;z) &\le -\alpha(1-\beta)\cos\lambda\left[\frac{n}{2} + (1-\beta)\cos^2\lambda\right]x^2\\ &\quad - [\alpha(1-\beta)^2\sin 2\lambda\cos\lambda]x\\ &\quad + \alpha\cos\lambda(\beta^2\cos^2\lambda - \sin^2\lambda)\\ &\quad + \alpha\beta\sin\lambda\sin 2\lambda + \beta(1-\alpha)\cos\lambda\\ &\quad - \frac{n\alpha}{2}(1-\beta).\end{aligned}$$

Equivalently, we have

$$\begin{aligned}\mathrm{Re}\,\phi(ix,y;z) &\le -Lx^2 - Mx + N\\ &= -\left[\sqrt{L}x + \frac{M}{2\sqrt{L}}\right]^2 + \frac{M^2}{4L} + N\\ &< \frac{M^2}{4L} + N,\end{aligned}$$

where L, M and N are given in the hypothesis.

Let $\Omega = \left\{\omega; \mathrm{Re}\,\omega > \frac{M^2}{4L} + N\right\}$.

Then $\phi(p(z),zp'(z);z) \in \Omega$ and $\phi(ix,y;z) \notin \Omega, \forall$ real x and $y \le -n(1+x^2)/2, z \in E$. By an application of Lemma 1.1, we obtain the required result.

By taking $\beta = 0, n = 1, \lambda = 0$ and $g(z) = \frac{z}{1-z}$ in Theorem 2.1, we get the result proved in [1].

Corollary 2.2. If $f(z) \in A$, satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > -\frac{\alpha}{2},\ z \in E, \alpha \ge 0,$$

then $f(z) \in S^*$.

If we take $\beta = \frac{\alpha}{2}, n = 1, \lambda = 0$ and $g(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following result proved in [1].

Corollary 2.3. If $f(z) \in A$, satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > -\frac{\alpha^2}{4}(1-\alpha),$$

then $f(z) \in S^*\left(\frac{\alpha}{2}\right)$.

If we take $\lambda = 0$ and $g(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the result proved in [3].

Corollary 2.4. If $f(z) \in A_n$, satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > \alpha\beta\left[\beta + \frac{n}{2} - 1\right] + \left[\beta - \frac{\alpha n}{2}\right],$$

$0 \le \alpha, \beta < 1$, then $f(z) \in S_n^*(\beta)$.

Theorem 2.5. Let $\alpha \ge 0, 0 \le \beta < 1$ and λ is real with $|\lambda| < \frac{\pi}{2}$. If $f(z) \in A_n$ satisfies

$$\mathrm{Re}\left\{e^{i\lambda}\frac{(f*g)(z)}{z}\left(\alpha\frac{z(f*g)'(z)}{(f*g)(z)}+1-\alpha\right)\right\} > \left[\beta - \frac{n\alpha}{2}(1-\beta)\right]\cos\lambda,$$

then

$$\mathrm{Re}\, e^{i\lambda}\frac{(f*g)(z)}{z} > \beta\cos\lambda.$$

Proof. Consider

$$e^{i\lambda}\frac{(f*g)(z)}{z} = [(1-\beta)p(z)+\beta]\cos\lambda + i\sin\lambda.$$

Taking logarithmic differentiation, we get

$$\alpha\frac{z(f*g)'(z)}{(f*g)(z)} + 1 - \alpha = \frac{[\alpha(1-\beta)\cos\lambda]zp'(z)}{[(1-\beta)p(z)+\beta]\cos\lambda + i\sin\lambda} + 1$$

So

$$e^{i\lambda}\frac{(f*g)(z)}{z}\left(\alpha\frac{z(f*g)'(z)}{(f*g)(z)}+1-\alpha\right)$$
$$=[\alpha(1-\beta)\cos\lambda]zp'(z)$$
$$+[(1-\beta)\cos\lambda]p(z)$$
$$+(\beta\cos\lambda+i\sin\lambda).$$
$$=\phi(p(z),zp'(z);z).$$

For all real x and y satisfying $y\le -n(1+x^2)/2$, we have

$$\phi(ix,y;z)=[\alpha(1-\beta)\cos\lambda]y+[(1-\beta)\cos\lambda](ix)$$
$$+(\beta\cos\lambda+i\sin\lambda).$$

Taking real part on both sides, we have

$$\operatorname{Re}\phi(ix,y;z)=[\alpha(1-\beta)\cos\lambda]y+\beta\cos\lambda$$
$$\le -\tfrac{1}{2}n(1+x^2)\alpha(1-\beta)\cos\lambda+\beta\cos\lambda$$
$$\le \beta\cos\lambda-\frac{n\alpha(1-\beta)\cos\lambda}{2}.$$

Let us take

Let $\Omega=\left\{\omega;\operatorname{Re}\omega>\left[\beta-\frac{n\alpha}{2}(1-\beta)\right]\cos\lambda\right\}$. Then $\phi(p(z),zp'(z);z)\in\Omega$ and $\phi(ix,y;z)\notin\Omega,\forall$ real x and $y\le -n(1+x^2)/2\,,z\in E$. By an application of Lemma 1.1, we obtain the required result.

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