Some approximation theorems via $\sigma$-convergence

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Abstract: The concept of $\sigma$-convergence was introduced in [P. Schaefer, Proc. Amer. Math. Soc. 36(1972)104-110] by using invariant mean. In this paper we apply this method to prove some Korovkin type approximation theorems.


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1. Introduction and preliminaries

Let $c$ and $\ell_\infty$ denote the spaces of all convergent and bounded sequences, respectively, and note that $c \subseteq \ell_\infty$. In the theory of sequence spaces, a beautiful application of the well known Hahn-Banach Extension Theorem gave rise to the concept of the Banach limit. That is, the limit functional defined on $c$ can be extended to the whole of $\ell_\infty$, and this extended functional is known as the Banach limit [2]. In 1948, Lorentz [8] used this notion of a weak limit to define a new type of convergence, known as the almost convergence. Later on, Raimi [17] gave a slight generalization of almost convergence and named it the $\sigma$-convergence. Before proceeding further, we should recall some notations and basic definitions used in this paper.

Let $\sigma$ be a mapping of the set of positive integers $\mathbb{N}$ into itself. A continuous linear functional $\varphi$ defined on the space $\ell_\infty$ of all bounded sequences is called an invariant mean (or a $\sigma$-mean; cf. [17]) if it is non-negative, normal and $\varphi(x) = \varphi \left( (x_\sigma(n)) \right)$. A sequence $x = x_k$ is said to be $\sigma$-convergent to the number $L$ if and only if all of its $\sigma$-means coincide with $L$, i.e. $\varphi(x) = L$ for all $\varphi$. A bounded sequence $x = x_k$ is $\sigma$-convergent (cf. [18]) to the number $L$ if and only if $\lim_{p \to \infty} t_{pm} = L$ uniformly in $m$, where

\[ t_{pm} = \frac{x_{m} + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^p(m)}}{p + 1} \]

We denote the set of all $\sigma$-convergent sequences by $V_\sigma$ and in this case we write $x_k \to L(V_\sigma)$ and $L$ is called the $\sigma$-limit of $x$. Note that a $\sigma$-mean extends the limit functional on $c$ in the sense that $\varphi = \lim x$ for all $x \in c$ if and only if $\sigma$ has no finite orbits (cf. [11, 12]) and $c \subseteq V_\sigma \subseteq \ell_\infty$.

If $\sigma$ is a translation then the $\sigma$-mean is called a Banach limit and $\sigma$-convergence is reduced to the concept of almost convergence introduced by Lorentz [8].

For $\sigma$-convergence of double sequences, refer the reader to [3, 12, 13, 14].

If $m = 1$ then we get $(C, 1)$-convergence, and in this case we write $x_k \to \ell(C, 1)$; where $\ell = (C, 1)$ - limit.

Remark 1.1. Note that:

(a) a convergent sequence is also $\sigma$-convergent;
(b) a $\sigma$-convergent sequence implies $(C, 1)$ convergent.

Example 1.2. The sequence $z = (z_n)$ defined as $z_n = \begin{cases} 1 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even} \end{cases}$ is $\sigma$-convergent to $1/2$ (for $\sigma(n) = n + 1$) but not convergent.

Let $C[a, b]$ be the space of all functions $f$ continuous on $[a, b]$. We know that $C[a, b]$ is a Banach space with norm $\|f\|_C := \sup_{a \leq x \leq b} |f(x)|$, $f \in C[a, b]$. Suppose that $T_n : C[a, b] \to C[a, b]$. We write $T_n f(x)$ for $T_n(f(t), x)$ and we say that $T$ is a positive operator if $T(f, x) \geq 0$ for all $f(x) \geq 0$.

The classical Korovkin approximation theorem states as follows [6, 7]:

Let $T_n$ be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$ and $\lim_{n} \|T_n f_i - f_i(x)\|_C = 0$, for $i = 0, 1, 2, \ldots$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$. Then $\lim_{n} \|T_n f(x) - f(x)\|_C = 0$, for all $f \in C[a, b]$.

Quite recently, such type of approximation theorems for functions of single variables were proved in [5, 9, 10, 15, 16] and for functions of two variables in [1, 4] by using statistical convergence and almost convergence. In this paper, we use the notion of $\sigma$-convergence to prove Korovkin type approximation theorems.

2. Korovkin type approximation theorem
The following is the $V_\sigma$-version of the classical Korovkin approximation theorem followed by an example to show its importance.

**Theorem 2.1.** Let $(T)_n$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ and $D_{n,p}(f,x) = \frac{1}{p} \sum_{k=1}^{p} T_{a^k_n}(f)(x)$ satisfying the following conditions

$$\lim_{p \to \infty} \|D_{n,p}(1, x) - 1\|_{\infty} = 0 \quad \text{uniformly in } n, \quad (2.1.1)$$

$$\lim_{p \to \infty} \|D_{n,p}(t, x) - x\|_{\infty} = 0 \quad \text{uniformly in } n, \quad (2.1.2)$$

$$\lim_{p \to \infty} \|D_{n,p}(t^2, x) - x^2\|_{\infty} = 0 \quad \text{uniformly in } n. \quad (2.1.3)$$

Then for any function $f \in C[a,b]$ bounded on the whole real line, we have

$$\sigma- \lim_{p \to \infty} \|T_k(f, x) - f(x)\|_{\infty} = 0 \quad \text{i.e.,}$$

$$\lim_{p \to \infty} \|D_{n,p}(f, x) - f(x)\|_{\infty} = 0 \quad \text{uniformly in } n,$$

Proof. Since $f \in C[a,b]$ and $f$ is bounded on the real line, we have

$$|f(x)| \leq M, \quad -\infty < x < \infty.$$

Therefore,

$$|f(t) - f(x)| \leq 2M, \quad -\infty < t, x < \infty \quad (2.1.4)$$

Also we have that $f$ is continuous on $[a,b]$, i.e.,

$$|f(t) - f(x)| \leq c, \quad \forall |t - x| < \delta \quad (2.1.5)$$

Using (2.1.4), (2.1.5) and putting $\psi(t) = (t - x)^2$, we get

$$|f(t) - f(x)| < \epsilon + \frac{2M}{\delta^2} \psi, \forall |t - x| < \delta,$$

This means

$$-\epsilon - \frac{2M}{\delta^2} \psi < f(t) - f(x) < \epsilon + \frac{2M}{\delta^2} \psi.$$

Now, we operating $T_{a^k_n}(1, x)$ for all $n$ to this inequality since $T_{a^k_n}(f, x)$ is monotone and linear. Hence

$$T_{a^k_n}(1, x) \left( -\epsilon - \frac{2M}{\delta^2} \psi \right) < T_{a^k_n}(1, x) \left( f(t) - f(x) \right) < T_{a^k_n}(1, x) \left( \epsilon + \frac{2M}{\delta^2} \psi \right).$$

Note that $x$ is fixed and so $f(x)$ is constant number. Therefore

$$-\epsilon T_{a^k_n}(1, x) - \frac{2M}{\delta^2} T_{a^k_n}(\psi, x) < T_{a^k_n}(f, x) - f(x) T_{a^k_n}(1, x) < \epsilon T_{a^k_n}(1, x) + \frac{2M}{\delta^2} T_{a^k_n}(\psi, x). \quad (2.1.6)$$

But

$$T_{a^k_n}(f, x) - f(x) = T_{a^k_n}(f, x) - f(x) T_{a^k_n}(1, x) - f(x)$$

$$= [T_{a^k_n}(f, x) - f(x) T_{a^k_n}(1, x)]$$

$$+ f(x) [T_{a^k_n}(1, x) - 1] \quad (2.1.7)$$

Using (2.1.6) and (2.1.7), we have

$$T_{a^k_n}(f, x) - f(x) < \epsilon T_{a^k_n}(1, x)$$

$$+ \frac{2M}{\delta^2} (T_{a^k_n}(\psi, x) + f(x) [T_{a^k_n}(1, x) - 1]) \quad (2.1.8)$$

Let us estimate $T_{a^k_n}(\psi, x)$

$$T_{a^k_n}(\psi, x) = T_{a^k_n}((t - x)^2, x)$$

$$= T_{a^k_n}(t^2 - 2tx + x^2, x)$$

$$= T_{a^k_n}(t^2, x) + 2x T_{a^k_n}(t, x) + x^2 T_{a^k_n}(1, x)$$

$$= [T_{a^k_n}(t^2, x) - x] - 2x [T_{a^k_n}(t, x) - x]$$

$$+ x^2 [T_{a^k_n}(1, x) - 1].$$

Using (2.1.8), we obtain

$$T_{a^k_n}(f, x) - f(x) < \epsilon T_{a^k_n}(1, x)$$

$$+ \frac{2M}{\delta^2} \left( [T_{a^k_n}(t^2, x) - x^2] + 2x [T_{a^k_n}(t, x) - x] + x^2 [T_{a^k_n}(1, x) - 1] \right)$$

$$+ f(x) [T_{a^k_n}(1, x) - 1]$$

$$+ \frac{2M}{\delta^2} \left( [T_{a^k_n}(t^2, x) - x^2] + 2x [T_{a^k_n}(t, x) - x] + x^2 [T_{a^k_n}(1, x) - 1] \right)$$

$$+ f(x) [T_{a^k_n}(1, x) - 1].$$

Since $\epsilon$ is arbitrary, we can write

$$T_{a^k_n}(f, x) - f(x) \leq \epsilon [T_{a^k_n}(1, x) - 1]$$

$$+ \frac{2M}{\delta^2} \left( [D_{n,p}(t^2, x) - x^2] + 2x [D_{n,p}(t, x) - x] + x^2 [D_{n,p}(1, x) - 1] \right)$$

$$+ f(x) [T_{a^k_n}(1, x) - 1].$$

Similarly

$$D_{n,p}(f, x) - f(x) \leq \epsilon [D_{n,p}(1, x) - 1]$$

$$+ \frac{2M}{\delta^2} \left( [D_{n,p}(t^2, x) - x^2] + 2x [D_{n,p}(t, x) - x] + x^2 [D_{n,p}(1, x) - 1] \right)$$

$$+ f(x) [D_{n,p}(1, x) - 1].$$

and therefore

$$\|D_{n,p}(f, x) - f(x)\|_{\infty} \leq \left( \epsilon + \frac{2M b^2}{\delta^2} + M \right)$$

$$\|D_{n,p}(1, x) - 1\|_{\infty} + \frac{4Mb}{\delta^2} \|D_{n,p}(t, x) - x\|_{\infty}$$

$$+ \frac{2M}{\delta^2} \|D_{n,p}(t^2, x) - x^2\|_{\infty}.$$
Letting $p \to \infty$ and using (2.1.1), (2.1.2), (2.1.3), we get
\[
\lim_{p \to \infty} \| D_{n,p}(f,x) - f(x) \|_\infty = 0 \text{ uniformly in } n
\]
This completes the proof of the theorem.

In the following we give an example of a sequence of positive linear operators satisfying the conditions of Theorem 2.1 but does not satisfy the conditions of the Korovkin theorem.

**Example 2.2.** Consider the sequence of classical Bernstein polynomials
\[
b_n(f,x) := \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k},
\]
\[0 \leq x \leq 1.\]

Let the sequence $(P_n)$ be defined by $P_n: C[0,1] \to C[0,1]$ with $P_n(f(x)) = (1 + z_n B_n(f,x)$, where $z_n$ is defined as in Example 1.2. Then
\[
B_n(1,x) = 1, B_n(t,x) = x, B_n(t^2,x) = x^2 + \frac{t^n}{n},
\]
and the sequence $(P_n)$ satisfies the conditions (2.1.1)-(2.1.3). Hence we have
\[
\sigma-\lim \| P_n(f,x) - f(x) \|_\infty = 0.
\]
On the other hand, we get $\| P_n(f,0) \| = (1 + z_n f(0)$, since $B_n(f,0) = f(0)$, and hence
\[
\| P_n(f,x) - f(x) \|_\infty \geq | P_n(f,0) | = z_n | f(0) |.
\]
We see that $(P_n)$ does not satisfy the classical Korovkin theorem, since $\lim \sup_{n \to \infty} z_n$ does not exists.

Now we present a slight general result.

**Theorem 2.3.** Let $T_n$ be a sequence of positive linear operators on $C[a,b]$ such that
\[
\lim_{n \to \infty} \| T_{n+1} - T_n \|_\infty = 0 \quad (2.3.1)
\]
If
\[
\sigma-\lim \| T_n(t^\nu - x) - x^\nu \|_\infty = 0 \quad (\nu = 0, 1, 2).
\]
Then for any function $f \in C[a,b]$ bounded on the real line, we have
\[
\lim_{n \to \infty} \| T_n(f,x) - f(x) \|_\infty = 0 \quad (2.3.3)
\]
**Proof.** From Theorem 2.1, we have that if (2.3.2) holds then
\[
\lim_{p \to \infty} \| D_{n,p}(f,x) - f(x) \|_\infty = 0, \text{ uniformly in } n \quad (2.3.4)
\]
We have the following inequality
\[
\| T_n(f,x) - f(x) \|_\infty \leq \| D_{n,p}(f,x) - f(x) \|_\infty + \frac{1}{p} \sum_{k=n+1}^{n+p-1} \left( \sum_{l=n+1}^k \| T_l - T_{l-1} \|_\infty \right)
\]
\[
\leq \| D_{n,p}(f,x) - f(x) \|_\infty + \frac{p-1}{2} \left( \sup_{k \geq n} \| T_k \| - \| T_{k-1} \|_\infty \right) \quad (2.3.5)
\]
Hence using (2.3.1) and (2.3.4), we get (2.3.3).

This completes the proof of the theorem.

**Remark 2.4.** We know that $\sigma$-convergence implies $(C,1)$ convergence. This motivates us to further generalize our main result by weakening the hypothesis or to add some condition to get more general result.

**Theorem 2.5.** Let $(T_n)$ be a sequence of positive linear operators on $C[a,b]$ such that
\[
(C,1) - \lim_{n \to \infty} \| T_n(t^\nu - x) - x^\nu \|_\infty = 0 \quad (\nu = 0, 1, 2) \quad (2.5.1)
\]
and
\[
\lim_{p \to \infty} \sup \left\{ \frac{n}{p} \left( \| \xi_{n+p-1}(f,x) - \xi_{n-1}(f,x) \|_\infty \right) \right\} = 0 \quad (2.5.2)
\]
where
\[
\xi_n(f,x) = \frac{1}{n+1} \sum_{k=0}^n T_k(f,x).
\]
Then for any function $f \in C[a,b]$ bounded on the real line, we have
\[
\sigma-\lim \| T_n(f,x) - f(x) \|_\infty = 0.
\]
**Proof.** For $n \geq p \geq 1$, it is easy to show that
\[
D_{n,p}(f,x) = \xi_{n+p-1}(f,x) + \frac{n}{p} \left( \xi_{n+p-1}(f,x) - \xi_{n-1}(f,x) \right),
\]
which implies
\[
\sup_{n \geq p} \| D_{n,p}(f,x) - \xi_{n+p-1}(f,x) \|_\infty = \sup_{n \geq p} \left\{ \frac{n}{p} \left( \| \xi_{n+p-1}(f,x) - \xi_{n-1}(f,x) \|_\infty \right) \right\} = \left( \| \xi_{n+p-1}(f,x) - \xi_{n-1}(f,x) \|_\infty \right) \quad (2.5.3)
\]
Also by Theorem 2.1, Condition (2.5.1) implies that
\[
(C,1) - \lim_{n \to \infty} \| T_n(f,x) - f(x) \|_\infty = 0 \quad (2.5.4)
\]
Using (2.5.1)-(2.5.4) and the fact that $\sigma$-convergence implies $(C,1)$ convergence, we get the desired result.

This completes the proof of the theorem.

**Theorem 2.6.** Let $(T_n)$ be a sequence of positive linear operators on $C[a,b]$ such that
\[
\lim \sup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \| T_n - T_{\sigma^k(m)} \| = 0.
\]
If
\[
\sigma-\lim \| T_n(t^\nu - x) - x^\nu \|_\infty = 0 \quad (\nu = 0, 1, 2) \quad (2.6.1)
\]
Then for any function $f \in C[a,b]$ bounded on the real line, we have
\[ \lim_{n} \| T_n(f,x) - f(x) \|_{\infty} = 0. \quad (2.6.2) \]

**Proof.** From Theorem 2.1, we have that if (2.6.1) holds then
\[ \sigma\text{-lim} \| T_n(f,x) - f(x) \|_{\infty} = 0, \]
which is equivalent to
\[ \lim_{n} \left\| \sup_{m} D_{m,n}(f,x) - f(x) \right\|_{\infty} = 0 \]
Now
\[ T_n - D_{m,n} = T_n - \frac{1}{n} \sum_{k=0}^{n-1} T_{\sigma^{k}(m)} = \frac{1}{n} \sum_{k=0}^{n-1} (T_n - T_{\sigma^{k}(m)}). \]
Therefore
\[ T_n - \sup_{m} D_{m,n} = \sup_{m} \frac{1}{n} \sum_{k=0}^{n-1} (T_n - T_{\sigma^{k}(m)}). \]
Hence using the hypothesis we get
\[ \lim_{n} \| T_n(f,x) - f(x) \|_{\infty} = \lim_{n} \left\| \sup_{m} D_{m,n}(f,x) - f(x) \right\|_{\infty} = 0, \]
that is (2.6.2) holds.

**References**


