A very short note on the best bounds in Sandor and Debnath's inequality

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Abstract: In this short note, we discuss the best bounds of the Sandor and Debnath's inequality and we obtain in simple proof that

$$\frac{e^{-x}\sqrt{2\,\pi}\,x^{x}}{\sqrt{x-(2\,\gamma-1)}}<\Gamma(x)<\frac{e^{-x}\sqrt{2\,\pi}\,x^{x}}{\sqrt{x-1/6}},\quad x>1$$

where γ is the Euler- Mascheroni constant.
[M. Mansour. **A very short note on the best bounds in Sandor and Debnath's inequality.** *Life Sci J* 2012;9(4):810-812] (ISSN:1097-8135). http://www.lifesciencesite.com. 126

Key Words: Stirling's formula, Sandor and Debnath's inequality, gamma function, digamma function, factorial *n*, best bounds.

MSC 2010 classification : 33B15, 26D07, 40A25.

1 Introduction.

Stirling's formula for factorials in its simplest form is

$$n!\sim\sqrt{2\,n\,\pi}\left(\frac{n}{e}\right)^{n} \qquad (1)$$

This approximation is used in many applications, especially in statistics and in the theory of probability to help estimate the value of $n!$, where $\sim$ is used to indicate that the ratio of the two sides goes to 1 as n goes to ∞. In other words, we have

$$\lim_{n\to\infty}\frac{n!}{n^{n+1/2}\,e^{-n}}=\sqrt{2\,\pi}.$$

Stirling's formula was actually discovered by De Moivre (1667-1754) but James Stirling (1692-1770) improved it by finding the value of the constant $\sqrt{2\,\pi}$. A number of upper and lower bounds for $n!$ have been obtained by various authors [4].

J. Sandor and L. Debnath [7] found the following double inequality

$$\frac{e^{-n}\sqrt{2\,\pi}\,n^{n+1}}{\sqrt{n}}<n!<\frac{e^{-n}\sqrt{2\,\pi}\,n^{n+1}}{\sqrt{n-1}}\quad n\geq 2 \qquad (2)$$

After that, this formula and other similar estimations were established by Guo [3]. N. Batir [1] refined and extended the double inequality (2) to the form for $n\geq 1$

$$\alpha_n=\frac{e^{-n}\sqrt{2\,\pi}\,n^{n+1}}{\sqrt{n-(1-2\pi e^{-2})}}<n!<\frac{e^{-n}\sqrt{2\,\pi}\,n^{n+1}}{\sqrt{n-1/6}}=\beta_n \quad (3)$$

which is better than the Burnside's formula for [2]

$$n!\sim\sqrt{2\,\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} \qquad (4).$$

C. Mortici [5] discuss in the double inequality (2) and established an asymptotic expansion, leading to a new accurate approximation formula which provides all exact digits of $n!$, for every $n\leq 28$. Mortici's formula is stronger than the upper bound β_n in the double inequality (3).

In this short note, we will improve the lower bound of the double inequality (3) and we will prove its upper bound by different method. Throughout this work, the logarithmic derivative of the gamma function

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt,$$

denoted by

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)},$$

is called the psi or digamma function. One of the elementary properties of the gamma function is the functional equation $\Gamma(x+1)=x\Gamma(x)$, in particular $n!=\Gamma(n+1)$.

In order to prove our main result we need the following Theorem

Theorem 1.
For $x>1$

$$\log x - \frac{1}{2x} - \frac{1}{12\,x^2} < \psi(x) < \log x - \frac{1}{2x} - \frac{2\gamma - 1}{2\,x^2}, \qquad (5)$$

where γ is the Euler-Mascheroni constant.

2 Main result

Our main result is the following Theorem:

Theorem 2.
For $x > 1$

$$\frac{e^{-x}\sqrt{2\,\pi}\,x^x}{\sqrt{x-(2\,\gamma-1)}} < \Gamma(x) < \frac{e^{-x}\sqrt{2\,\pi}\,x^x}{\sqrt{x-1/6}}, \qquad (6)$$

where γ is the Euler-Mascheroni constant.

Proof.
Let the function

$$M_\theta(x) = \frac{e^{-x}\sqrt{2\,\pi}\,x^x}{\sqrt{x-\theta}}\Gamma(x), \quad x > \theta > 0. \qquad (7)$$

It is clear that

$$M_{\theta_1}(x) < M_{\theta_2}(x), \quad \forall \theta_1 < \theta_2, \qquad (8)$$

which means that $M_\theta(x)$ is increasing function w.r.t. θ. Also,

$$\lim_{x\to\infty} M_\theta(x) = 1. \qquad (9)$$

Now

$$\frac{d}{dx}M_\theta(x) = M_\theta(x)\left(\frac{-1}{2(x-\theta)} + \log x - \psi(x)\right) \qquad (10)$$

There are two cases:
The first case if we take

$$\log x - \frac{1}{2x} - \frac{1}{12\,x^2} < \psi(x),$$

then we get for $x > 1$, β that

$$\begin{aligned}\frac{d}{dx}M_\beta(x) &< M_\beta(x)\left(\frac{-1}{2(x-\beta)} + \frac{1}{2x} + \frac{1}{12\,x^2}\right)\\ &< M_\beta(x)\left(\frac{x-(1+6x)\beta}{12\,x^2(x-\beta)}\right)\\ &< 0\end{aligned}$$

if $x-(1+6x)\beta \le 0$, which satisfies if $\beta \ge 1/6$. Then the function $M_\beta(x)$ is decreasing function for $\beta \ge 1/6$ and $x > 1$. But

$$\lim_{x\to\infty} M_\beta(x) = 1,$$

then we obtain

$$M_\beta(x) > 1, \qquad \beta \ge \frac{1}{6}; x > 1 \qquad (11)$$

Also, $M_\beta(x)$ is increasing function w.r.t. β, then

$$M_\beta(x) > M_{1/6}(x) > 1, \qquad \beta > \frac{1}{6}; x > 1$$

which give us the following best upper bound of Sandor and Debnath's inequality

$$\Gamma(x) < \frac{e^{-x}\sqrt{2\,\pi}\,x^x}{\sqrt{x-1/6}}, \quad x > 1. \qquad (12)$$

The second case if we take

$$\psi(x) < \log x - \frac{1}{2x} - \frac{2\gamma-1}{2\,x^2},$$

then we get for $x > 1$, μ that

$$\begin{aligned}\frac{d}{dx}M_\mu(x) &> M_\mu(x)\left(\frac{-1}{2(x-\mu)} + \frac{1}{2x} + \frac{2\gamma-1}{2\,x^2}\right)\\ &> M_\mu(x)\left(\frac{(x-\mu)(2\gamma-1)-\mu x}{2\,x^2(x-\mu)}\right)\\ &> 0\end{aligned}$$

if $(x-\mu)(2\gamma-1)-\mu x \ge 0$, which equivalent

$$\mu \le \frac{x}{1+\frac{x}{2\gamma-1}} \le 2\gamma - 1 \quad \forall\, x > 1.$$

Then the function $M_\mu(x)$ is increasing function for $\mu \le 2\gamma - 1$ and $x > 1$. But

$$\lim_{x\to\infty} M_\mu(x) = 1,$$

then we obtain

$$M_\mu(x) < 1, \qquad \mu \le 2\gamma - 1\,; x > 1. \quad (13)$$

Also, $M_\mu(x)$ is increasing function w.r.t. μ, then

$$M_\mu(x) \le M_{2\gamma-1}(x) < 1, \qquad \mu \le 2\gamma - 1\,; x > 1$$

which give us the following best lower bound of Sandor and Debnath's inequality

$$\Gamma(x) > \frac{e^{-x}\sqrt{2\,\pi}\,x^x}{\sqrt{x-(2\,\gamma-1)}}, \qquad x > 1. \qquad (14)$$

In particular, if we put $x = 1$ in (6), we have for $n > 1$

$$\mu_n = \frac{e^{-n}\sqrt{2\,\pi}\,n^{n+1}}{\sqrt{n-(2\,\gamma-1)}} < n! < \frac{e^{-n}\sqrt{2\,\pi}\,n^{n+1}}{\sqrt{n-1/6}} = \beta_n \qquad (15)$$

It is clear that $1 - 2\pi e^{-2} < 2\,\gamma - 1$, which gives us that $\alpha_n < \mu_n < n!$ for $n > 1$. Then the lower bound of (15) is better than the lower bound of (3).

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9/18/2012