On the solution of a functional integral equation of Fredholm type with degenerate kerenel

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Abstract: Here, the existence and the uniqueness of the solution of a class of a nonlinear integral equation with discontinuous kernel are discussed and proved. A degenerate kernel method is used, as a numerical method, to obtain a class of a system of a nonlinear algebraic equations. Many important theorems related to the existence and uniqueness of the produced algebraic system are derived. Finally, numerical examples are discussed and the error estimate, in each case, is calculated.

[Abdallah A. Badr. **On the solution of a functional integral equation of Fredholm type with degenerate kerenel**. *Life Sci J* 2012;9(3):2127-2130] (ISSN:1097-8135). http://www.lifesciencesite.com. 307

Keywords: Nonlinear integral equation (NIE), nonlinear algebraic system (NAS), degenerate kernel method, Hammerstein integral equation.

1. Introduction

Integral equations of various types and kinds play an important role in many branches of linear and nonlinear functionals analysis and their applications in the life science, mathematical physics, engineering and contact problems in the theory of elasticity (see [1-3]). Therefore many different methods and numerical treatments are established to obtain the solution of the NIE. For these methods see Brunner et al. [4], Kaneko and Xu [5], Kilbas and Saigo [6], Dariusz [7], Abdou et al. [8,9], and Diogo and Lima [10]. This paper is concerned with finding a numerical solution of the following functional integral equation

$$\mu\phi(x)-\lambda f(x,\int_0^1 k(x,y)\gamma(y,\phi(y))dy)=g(x) \qquad (1)$$

where f, k, γ and g are known continuous functions while ϕ is unknown function, μ is a constant determine the kind of the IE and λ is a constant, may be complex, and has many physical meanings. The importance of Eq.(1) comes from it's special cases, for example when $f(x,u(x))=u(x)$, we have

$$\mu\phi(x)-\lambda\int_0^1 k(x,y)\gamma(y,\phi(y))dy=g(x).$$

This equation is called a Hammerstein integral equations.

In this work, the existence and uniqueness solution of Eq.(1), under certain conditions, are discussed and proved. Also, we present the degenerate kernel method and we consider the problem of the existence and uniqueness of the solution of the new NAS associated with the degenerate kernel. Also, the convergence problem of the numerical solution is also considered. Many examples are presented and the error estimate, in each case, is computed.

2. The existence and uniqueness solution

In order to guarantee the existence of a unique solution to Eq.(1), we will assume throughout this work the following conditions:

(i) The kernel $k(x,y)$ and the given function $g(x)$ are in the class $C([0,1]\times[0,1])$ and satisfies, in general the condition

$$\{\int_0^1\int_0^1 k^2(x,y)dxdy\}^{1/2}\le A,$$

$$\| g(x)\|_{L_2[0,1]}=\{\int_0^1 g^2(x)dx\}^{1/2}=B.$$

(ii) The two continuous functions $f(x,u(x))$ and $\gamma(x,v(x))$, where $x\in[0,1]$, and $u,v\in(-\infty,\infty)$ satisfy the condition

$$\{\int_0^1 | f^2(x,u(x))|^2\, dx\}^{1/2}\le C_1\| u\|,$$

$$\{\int_0^1 |\gamma(x,v(x))|^2\, dx\}^{1/2}\le C_2\| v\|$$

(C_1, C_2 are constants).

(iii) The two functions $f(x,u(x))$ and $\gamma(x,v(x))$ satisfy Lipschitz condition for the second argument

$$| f(x,u_1(x))-f(x,u_2(x))|\le D_1 | u_1(x)-u_2(x)|,$$

$$|\gamma(x,v_1(x))-\gamma(x,v_2(x))|\le D_2 | v_1(x)-v_2(x)|$$

(D_1 and D_2 are constants).

Theorem (1): Under the following condition

$$|\lambda|<\frac{|\mu|}{AD_1D_2}$$

the NIE (1) has a unique solution in $L_2[0,1]$ where the radius of convergence is given by

$$\rho = \frac{B}{(|\mu| - |\lambda| A D_1 D_2)}.$$

This can be proved by a direct application to the Banach contraction principal. To obtain a higher order convergence rate, we need to assume higher order smoothness conditions on the kernel $k(x,y)$.

3. Degenerate kernel method

Suppose that $k_n(x,y)$ is an approximation of the kernel $k(x,y)$ and that it is of the degenerate form

$$k_n(x,y) = \sum_{i=1}^{n} B_i(x) C_i(y) \qquad (2)$$

where $\{B_i(x)\}$ and $\{C_i(y)\}$ are assumed to be a linearly independent set of functions in $L_2[0,1]$. Also, we assume

$$\{\int_0^1\int_0^1 |k(x,y) - k_n(x,y)|^2\, dxdy\}^{1/2} \to 0,\ n \to \infty. \qquad (3)$$

Hence, the expected solution of the NIE associated with the degenerate kernels $k_n(x,y)$ which converges to the exact solution of Eq.(1) is of the form

$$\mu\,\phi_n(x) - \lambda f(x, \int_0^1 k_n(x,y)\gamma(y,\phi_n(y))dy) = g(x). \qquad (4)$$

To obtain the solution of this equation, $\phi_n(x)$, we use (2), in this equation to get

$$\mu\,\phi_n(x) = g(x) + \lambda f(x, \sum_{i=1}^{n} \alpha_{i,n} B_i(x)), \qquad (5)$$

where

$$\alpha_{i,n} = \int_0^1 C_i(y)\gamma(y,\phi_n(y))dy, \quad 1 \le i \le n. \qquad (6)$$

Once the constants $\alpha_{i,n}$ have been determined, the approximate solutions of (5) are obtained.

Substituting (5) into (6), we have

$$\alpha_{j,n} = \int_0^1 C_j(y)\gamma[y, \frac{g(y)}{\mu} + \frac{\lambda}{\mu} f(y, \sum_{i=1}^{n} \alpha_{i,n} B_i(y))]dy. \qquad (7)$$

Define

$$H_j(\alpha_1,\alpha_2,,\alpha_n) = \int_0^1 C_j(y)\gamma[y, \frac{g(y)}{(\mu)} + \frac{\lambda}{\mu} f(y, \sum_{i=1}^{n} \alpha_{i,n} B_i(y))]dy$$

(8)

Then, the formula (7) represents a NAS, which can be written in a vector notation as

$$\alpha = F(\alpha) \qquad (9)$$

where $\alpha^T = (\alpha_1, \alpha_2, , \alpha_n)$ and $F^T(\alpha) = (F_1(\alpha), F_2(\alpha), ..., F_n(\alpha))$.

In other words, the numerical solution of the NIE (1) reduces to an optimization problem in which an unknown scalar vector α is to be found such that

$\alpha - F(\alpha)$ is minimized.

4.Nonlinear algebraic system

Now, we shall show that, under some mild assumptions, the unique solution of the NAS (9) corresponds to the unique solution of Eq.(5) for each $n, n = 1,2,3,..$

To prove that the NIE (5) has a unique solution in $L_2[0,1]$, we write Eq.(5) in the integral operator form

$$(\overline{W}_n\phi)(x) = \frac{g(x)}{\mu} + (W_n\phi)(x), \quad (\mu \ne 0), \qquad (10)$$

(2) where

$$(W_n\phi)(x) = \frac{\lambda}{\mu} f(x, \int_0^1 k_n(x,y)\gamma(y,\phi(y))dy). \qquad (11)$$

Also, in view of conditions (i) and (iii) there exists an integer N such that for each $n > N$, and after neglecting a very small constant, we have

$$\{\int_0^1\int_0^1 |k_n(x,y)|^2\, dx\, dy\}^{1/2} \le A. \qquad (12)$$

Theorem (2): Under the conditions (ii), (iii) and (12), the NIE (5) has a unique solution.

The proof of this theorem depends on the following two lemmas.

Lemma 1: Under the conditions (12) and (ii) the operator $\overline{W}_n$ defined by (10) maps the space $L_2[0,1]$ into itself.

Proof: In view of the formulas (11) and (10), we get

$$\|\overline{W}_n\phi\|_{L_2[0,1]} \le \frac{1}{|\mu|}\|g(x)\| + \left|\frac{\lambda}{\mu}\right| \| f(x, \int_0^1 k_n(x,y)\gamma(y,\phi(y))dy)\|. \qquad (13)$$

Applying Cauchy-Schwarz inequality, then using the conditions (12) and (ii), the above inequality can be adapted to

$$\|\overline{W}_n\phi\|_{L_2[0,1]} \le \frac{B}{|\mu|} + \sigma_1\|\phi\|, \quad (\sigma_1 = \frac{\lambda}{|\mu|} C C_1 C_2 > 1). \qquad (14)$$

The last inequality shows that, the operator $\overline{W}_n$ maps the ball S_{ρ_1} into itself where

$$\rho_1 = \frac{B|\mu|}{(|\mu| - |\lambda| C C_1 C_2)}. \qquad (15)$$

Moreover, the inequality (14) involves the boundness of the operator W and $\overline{W}$ given by Eq.(11) and of Eq.(10) respectively.

lemma 2 : Under the conditions (12), (ii) and (iii) the operator $\overline{W}_n$ is continuous in the space $L_2[0,1]$.

Proof : For two functions ϕ_1 and ϕ_2 in $L_2[0,1]$, we have

$$\|\overline{W}_n\phi_1-\overline{W}_n\phi_2\|\le|\frac{\lambda}{\mu}|.\|f(x,\int_0^1 k_n(x,y)\gamma(y,\phi_1(y))dy)$$
$$-f(x,\int_0^1 k_n(x,y)\gamma(y,\phi_2(y))dy)\|.$$

Applying Cauchy-Schwarz inequality, and with the aid of conditions (12) and (iii), we get

$$\|\overline{W}_n\phi_1-\overline{W}_n\phi_2\|\le|\frac{\lambda}{\mu}|CD_1D_2\|\phi_1-\phi_2\|. \quad (16)$$

This inequality shows that, the operator $\overline{W}$ is continuous in the space $L_2[0,1]$. Moreover under the condition $|\lambda|<\frac{|\mu|}{CD_1D_2}$, the operator $\overline{W}_n$ is a contractive in the space $L_2[0,1]$. Then by Banach fixed point theorem, the operator $\overline{W}_n$ has a unique fixed point which is, of course, the unique solution of Eq. (5) .

Now, we go to prove that the NAS of Eq.(9) has a unique solution. And this solution is the same solution of Eq. (5).

Theorem (3) : Under the condition

$$G=B_1D_1\{\sum_{i=1}^{n}\int_0^1|B_i(x)|^2\,dx\}^{1/2}\{\sum_{i=1}^{n}\int_0^1|C_i(x)|^2\,dx\}^{1/2}<1. \quad (17)$$

The NAS (9) have a unique solution $\alpha^*=(\alpha^*_{1n},\alpha^*_{2n},..,\alpha^*_{nn})$ and

$$\mu\phi_n(x)=g(x)+\lambda f(x,\sum_{i=1}^{n}\alpha^*_{in}B_i(x)) \quad (18)$$

is the unique solution of Eq. (5).

Proof: Define the discrete ℓ_2 norm by $\|\alpha\|_{\ell_2}=\{\sum_{i=1}^{n}|\alpha_i|^2\}^{1/2}$

for $\alpha=(\alpha_1,\alpha_2,.,\alpha_n)^T\in\ell_2(n)$.

Then for $\alpha^{(1)}=(\alpha^{(1)}_1,\alpha^{(1)}_2,,\alpha^{(1)}_n)$ and $\alpha^{(2)}=(\alpha^{(2)}_1,\alpha^{(2)}_2,,\alpha^{(2)}_n)$, using Eq.(8), we have

$$\|H(\alpha^{(1)})-H(\alpha^{(2)}\|_{\ell_2}=$$
$$\|\int_0^1 C_i(y)\{\gamma[y,g(y)+f(y,\sum_{i=1}^{n}\alpha^{(1)}_{in}B_i(y)]$$
$$-\gamma[y,g(y)+f(y,\sum_{i=1}^{n}\alpha^{(2)}_{in}B_i(y)]\}dy\|_{\ell_2}.$$

Using the condition (iii) on γ, we follow

$$\|H(\alpha^{(1)})-H(\alpha^{(2)}\|_{ell_2}\le$$
$$(\sum_{i=1}^{n}\int_0^1|C_i(x)|^2\,dx)^{1/2}B_1\|\int_0^1\{f(y,\sum_{i=1}^{n}\alpha^{(1)}_{in}B_i(y))$$
$$-f(y,\sum_{i=1}^{n}\alpha^{(2)}_{in}B_i(y)dy\|_{\ell_2}.$$

Finally, we have

$$\|H(\alpha^{(1)}-H(\alpha^2)\|\le B_1D_1(\sum_{i=1}^{n}\int_0^1|C_i(x)|^2\,dx)^{1/2}$$
$$(\sum_{i=1}^{n}\int_0^1|B_i(x)|^2\,dx)^{1/2}\|\alpha^{(1)}-\alpha^{(2)}\|_{\ell_2}.$$

Using the condition (17), H is a contraction operator in $\ell_2(n)$. There fore H has a unique fixed point α^*, i.e $\alpha^*=H(\alpha^*)$. For this α^*, it is obvious that $\phi_n(x)$ defined by (18) is a solution of (9), and by Theorem (2), $\phi_n(x)$ is the unique solution of (5).

5.The convergence

In this section, we study the rate of convergence of the the approximate solution $\phi_n(x)$ to the solution of Eq.(1), $\phi(x)$.

Theorem (4) : If condition (3) holds and if

$$\|k-k_n\|_{L_2[0,1]}=\{\int_0^1\int_0^1|k(x,y)-k_n(x,y)|^2\,dxdy\}^{1/2} \quad (19)$$

then we have

$$\|\phi-\phi_n\|_{L_2}\le\frac{|\lambda|D_1C_2\|\phi\|}{|\mu|-|\lambda|CD_1D_2}\|k-k_n\|. \quad (20)$$

Proof : We can write

$$\|\phi-\phi_n\|\le|\frac{\lambda}{\mu}|\{\int_0^1|f(x,\int_0^1 k(x,y)\gamma(y,\phi(y))dy)$$
$$-f(x,\int_0^1 k_n(x,y)\gamma(y,\phi_n(y))dy|^2\,dx\}^{1/2}. \quad (21)$$

Using condition (iv) and the inequality properties, we obtain

$$\|\phi-\phi_n\| \le \frac{\lambda}{\mu} | D_1 \{[\int_0^1\int_0^1 | k(x,y)-k_n(x,y)|^2 \, dxdy \int_0^1 |\gamma(y,\phi(y)|^2 \, dy]^{1/2} + [\int_0^1\int_0^1 |k_n(x,y)|^2 \, dxdy \int_0^1 |\gamma(y,\phi(y))-\gamma(y_1\phi_n(y))|^2 \, dy]^{1/2}\}.$$

Using the conditions (12), (iii) and (iv) and with the aid of Eq. (19), we have

$$\|\phi-\phi_n\| \le \frac{|\lambda| D_1 C_2 \|\phi\|}{|\mu|-|\lambda| C D_1 D_2} \|k-k_n\|.$$

6. Examples

Example 1:

Consider the integral equation

$$\phi(x)-\{\int_0^1 x^2 y \phi^2(y)dy\}^{\frac{1}{2}} = \frac{x}{2}.$$

Assume $B_1(x)=x^2, C_1(y)=y$. Hence Eq.(5) gives an approximate solution of the above equation in the form

$$\phi^* = \frac{x}{2} + x\sqrt{\alpha}.$$

Using Eq.(6), the parameter α is given by

$$\alpha = \int_0^1 y[\frac{y}{2}+y\sqrt{\alpha}]^2 dy.$$

Solving this equation, we have $\alpha=\frac{1}{4}$ or $\alpha=\frac{1}{36}$ and hence the approximate solution is $\phi^*(x)=x$ or $\phi^*(x)=\frac{2x}{3}$.

Example 2:

Consider the integral equation

$$\phi(x)-\{\int_0^1 (1+xy)\phi^2(y)dy\}^2 = \frac{5x}{6}-\frac{x^2}{16}-\frac{1}{9}.$$

Assume $B_1(x)=1, B_2(x)=x, C_1(y)=1, C_1(y)=1, C_2(y)=y$. Hence, Eq. (5) gives

$$\phi^*(x) = \frac{5x}{6}-\frac{x^2}{16}-\frac{1}{9}+(\alpha_1+\alpha_2 x)^2.$$

Using Eq.(6), the parameters α_1, α_2 are given by

$$\alpha_1 = \int_0^1 \{g(x)+[\alpha_1+\alpha_2 * B_2(x)]^2\}^2 dx,$$

$$\alpha_2 = \int_0^1 x\{g(x)+(\alpha_1+\alpha_2 * B_2(x))^2\}^2 dx.$$

Solving these equations, we obtain $\{\alpha_1=0.2206977763, \alpha_2=0.1518207037\}$ or $\{\alpha_1=0.3390085493, \alpha_2=0.2128606384\}$.

7. Discussions

We see from this paper that the numerical solution of the functional integral equation, Eq. (1) reduces to an optimization problem, Eq.(9). Once we obtain the solution of this optimization problem, an approximate solution to the functional integral is obtained. In practice, searching for this optimal solution is not an easy question. In example one, we found two solutions in which one of these two solutions gives us the exact solution. In example 2, we obtained a system of two algebraic equations each is of order four. One of this four solutions gives us a very good approximate to the exact . Among of the these solution, we select the solution in which $\alpha - F(\alpha)$ is minimized.

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