On some lower bounds and approximation formulas for $n!$

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Abstract: In this paper, we present the following new inequality of $n!$ $n! > \sqrt{2\pi}\ n^{n+1/2} e^{-n+\sum_{r=0}^{\infty}\left\{(2n+2r+1)\tanh^{-1}\left(\frac{1}{2n+2r+1}\right)-1\right\}}$ $n \in \mathbb{N}$. Also, we deduce that the approximation formula $n! \sim \sqrt{2\pi}\, n^{n+1/2}\ e^{-n+\sum_{k=1}^{m}\frac{2^{-2k}}{2k+1}\varsigma(2k,n+1/2)}$ has rate of convergence equal to n^{-2m-1} for $m = 1, 2, 3, \cdots$. Thus, we can choose the approximation formula that we want it convergence to $n!$ by a known rate.
[Mustafa A. OBAID. **On some lower bounds and approximation formulas for $n!$** *Life Sci J* 2012;9(3):743-] (ISSN:1097-8135). http://www.lifesciencesite.com. 105

Key Words: Bounds of $n!$ rate of convergence, approximation formulas.

MSC 2010 classification : 33B15, 26D07, 41A60.

1 Introduction.

There are many different upper and lower bounds for $n!$ presented by several authors [4, 3, 21, 20, 17, 8, 9]. Most bounds are of the form

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{a_n} < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{b_n}, \qquad (1)$$

Where a_n and b_n tend to zero through positive values. P. R. Beesack [2] presented the following important result:

Theorem 1.

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{a_n} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{b_n}$$

$$n \geq 1, \qquad (2)$$

where the two sequences a_n, $b_n \to 0$ as $n \to \infty$ and satisfy

$$a_n - a_{n+1} < \sum_{k=1}^{\infty}\frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} < b_n - b_{n+1}. \qquad (3)$$

For the q-factorial which is defined by [5]

$$[n]_q! = n_q[n-1]_q \cdots [2]_q[1]_q,$$

where $[x]_q = \frac{1-q^x}{1-q}$ is the q-number of x, Mansour and et al [6] presented the following q-analog of the Beesack's result (2):

Theorem 2. The q- factorial $[n]_q!$ satisfies the double inequality

$$(q,q)_\infty(1-q)^{-n}e^{f_q(n+1)} < [n]_q!\,(q;q)_\infty(1-q)^{-n}\ e^{g_q(n+1)}, \quad n \geq 1; 0 < q < 1 \qquad (4)$$

where $f_q(n)$ and $g_q(n)$ are two sequences tend to zero through positive values and satisfy

$$f_q(n) - f_q(n+1) - \log(1-q^n) < g_q(n) - g_q(n+1), \qquad n \geq 1. \qquad (5)$$

Recently, Mansour and et al [7] presented a new proof of Beesack's result (2) and deduced the following upper bounds of $n!$:

Theorem 3.

$$n! < \sqrt{2\pi n}\,(n/e)^n\, e^{M_n^{[m]}} \qquad n \in \mathbb{N} \qquad (6)$$

$$M_n^{[m]} = \frac{1}{2m+3}\left[\frac{1}{4n} + \sum_{k=1}^{m}\frac{2m-2k+2}{2k+1}\, 2^{-2k}\varsigma\left(2k, n+\frac{1}{2}\right)\right]$$

$$m = 1,2,3,\cdots,$$

where $\varsigma(x)$ is the Riemann Zeta function.

In this paper, we will use the technique of [7] to introduce a family of lower bounds of $n!$. Hence, we will deduce some new approximation formulas for large $n!$ and we will study their rates of convergence.

2 A New family of lower bounds of $n!$

To find some lower bounds of the series $\sum_{k=1}^{\infty}\frac{1}{2k+1}\frac{1}{(2n+1)^{2k}}$ we observe firstly that

$$\sum_{k=1}^{\infty}\frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} > \sum_{k=1}^{m}\frac{1}{(2k+1)(2n+1)^{2k}}, \quad m = 1,2,3,\cdots$$

So, we can consider the recurrence relation

$$L_{n,m} - L_{n+1,m} = \sum_{k=1}^{m}\frac{1}{(2k+1)(2n+1)^{2k}}, \quad m = 1,2,3,\cdots \qquad (7)$$

which has the following solution form

$$L_{n,m} = L_{0,m} - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{m}\frac{1}{(2k+1)(2i+1)^{2k}}\right)$$
$$= L_{0,m} - \sum_{k=1}^{m}\frac{1}{2k+1}\left(\sum_{i=1}^{n-1}\frac{1}{(2i+1)^{2k}}\right).$$

By using the relation [18]

$$\sum_{i=1}^{n-1}\frac{1}{(2i+1)^{2k}} = -1-(2^{-2k}-1)\varsigma(2k) - 2^{-2k}\varsigma(2k,n+1/2)$$
$$= -1-\frac{(-1)^{k-1}(1-2^{2k})}{2(2k)!}B_{2k}\pi^{2k}+2^{-2k}\varsigma(2k,n+1/2)$$

where $\varsigma(x)$ is the Riemann Zeta function and $B_r's$ are Bernoulli's numbers, we get

$$L_{n,m} = L_{0,m} + \sum_{k=1}^{m}\frac{1}{2k+1}\left(1 + \frac{(-1)^{k-1}(1-2^{2k})}{2(2k)!}B_{2k}\pi^{2k} + 2^{-2k}\varsigma(2k,n+1/2)\right).$$

Also

$$\sum_{i=1}^{\infty}\frac{1}{(2i+1)^{2k}} = \frac{(-1)^{k-1}(2^{2k}-1)}{2(2k)!}B_{2k}\pi^{2k}-1.$$

Hence, we can choose

$$L_{0,m} = \sum_{k=1}^{m}\frac{1}{2k+1}(\varsigma(2k)(1-2^{-2k})-1), \quad (8)$$

which satisfies

$$\lim_{n\to\infty} L_{n,m} = 0, \quad m = 1,2,3,\cdots.$$

Then we obtain the following result:

Theorem 4.

$$n! > \sqrt{2\pi}\; n^{n+\frac{1}{2}}\, e^{-n+\sum_{k=1}^{m}\frac{2^{-2k}}{2k+1}\varsigma\left(2k.n+\frac{1}{2}\right)}$$

$n, m \in \mathbb{N}$ (9)

where $\varsigma(x)$ is the Riemann Zeta function.

In the following result, we will prove that the increasing of the value of m in the lower bound $L_{n,m}$ will improve its value.

Lemma 2.1.

$L_{n,m+1} > L_{n,m} \qquad m, n = 1,2,3,\cdots.$ (10)

Proof.

From [9] we get

$$L_{n,m+1} = \sum_{k=1}^{m+1}\frac{2^{-2k}}{2k+1}\varsigma(2k,\frac{n+1}{2})$$
$$= L_{n,m} + \frac{2^{-2m-2}}{2m+3}\varsigma(2m+2,n+1/2).$$

But $\varsigma\left(2m+2,n+\frac{1}{2}\right) > 0$, then

$$L_{n,m+1} - L_{n,m} > 0.$$

Theorem 5.

$$n! > \sqrt{2\pi}\, n^{n+1/2}\, e^{-n+\sum_{r=0}^{\infty}\left\{(2n+2r+1)\tanh^{-1}\left(\frac{1}{2n+2r+1}\right)-1\right\}} \quad n \in \mathbb{N} \; (11)$$

Proof. Using (9) at m tends to ∞, we obtain

$$L_{n,\infty} = \sum_{k=1}^{\infty}\frac{2^{-2k}}{2k+1}\varsigma(2k,n+1/2).$$

But

$$\varsigma(2k,n+1/2) = \sum_{r=0}^{\infty}\frac{1}{(n+1/2\;+r)^{2k}},$$

then

$$L_{n,\infty} = \sum_{r=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2n+2r+1^{2k}(2k+1)}.$$

Using the relation

$$\tanh^{-1}x = \sum_{t=0}^{\infty}\frac{x^{2t+1}}{2t+1}; \qquad |x|<1,$$

then we get

$$L_{n,\infty} = \sum_{r=0}^{\infty}\left\{\begin{matrix}(2n+2r+1)\tanh^{-1}\\ ((2n+2r+1)^{-1})-1\end{matrix}\right\}.$$

3 Convergence rate of the approximation formula $n! \sim \sqrt{2\pi}\, n^{n+1/2} e^{-n+\sum_{k=1}^{m}\frac{2^{-2k}}{2k+1}\varsigma(2k,n=1/2)}$.

C. Mortici [10]—[16] presented a new method to measure the convergence rate of some asymptotic expansions. Also, he use this method to accelerate and construct some approximation formulas. The following lemma contains the Mortici result.

Lemma 3.1.

If $(\varphi_n)_{n\geq 1}$ is convergent to zero and there exists the limit

$$\lim_{n\to\infty} n^k(\varphi_n-\varphi_{n+1}) = l \in \mathbb{R} \qquad (12)$$

with $k > 1$, then there exists the limit:

$$\lim_{n\to\infty} n^{k-1}\varphi_n = \frac{l}{k-1}$$

To measure the convergence rate of the formula $\sqrt{2\pi n}\,(n/e)^n e^{L_{n,m}}$, define the sequence $(\varphi_n)_{n\geq 1}$ by the relation

$n! = \sqrt{2\pi n}\,(n/e)^n e^{L_{n,m}+\varphi_n}; \quad n = 1,2,3,\cdots$ (13)

The value of the approximation formula will be better whenever $(\varphi_n)_{n\geq 1}$ convergence to zero faster. Using the relation (13) we get

$\varphi_n = \ln n! - \ln\sqrt{2\pi} - (n+1/2)\ln n + n - L_{n,m}$

And hence

$\varphi_n - \varphi_{n+1} = (n+1/2)\ln(1+1/n) - 1 + L_{n+1,m} - L_{n,m}.$

By using the expansion [1]

$(n+1/2)\ln\left(1+\frac{1}{n}\right) - 1 = \sum_{k=1}^{\infty}\frac{1}{2k+1}\frac{1}{(2n+1)^{2k}}$ (14)

and the relation [7], we have

$$\varphi_n - \varphi_{n+1} = \sum_{k=m+1}^{\infty} \frac{1}{(2k+1)(2n+1)^{2k}}.$$

Then

$$\lim_{n\to\infty} n^{2(m+1)}(\varphi_n - \varphi_{n+1}) = \frac{1}{(2m+3)2^{2(m+1)}}; \quad n, m = 1,2,3,\cdots \quad (15)$$

Now we get the following result according Mortici result:

Theorem 6. The rate of convergence of the sequence φ_n is equal to n^{-2m-1} , since

$$\lim_{n\to\infty} n^{2m+1}\varphi_n = \frac{1}{(2m+1)(2m+3)2^{2(m+1)}}.$$

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6/23/2012