Reliability Equivalence of Independent Non-identical Parallel and Series Systems

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Abstract: The reliability equivalence factors of parallel and series systems with n independent non-identical components are obtained. Three different methods are used to improve such systems: (i) improving the quality of several components by decreasing their failure rates, (ii) adding a hot component to the system, and (iii) adding a cold redundant component to the system. The survival function is used as a performance measure of the system reliability to compare different system designs. The *rth* moment time to failures will be derived in parallel and series systems in Weibull Distribution.

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1. Introduction

Reliability evaluation is an important and integral feature of the planning, design and operation of all engineering systems. The failure of mechanical devices such as ships, trains, and cars, is similar in many ways to the life or death of biological organisms. A reliability equivalence factor is a factor by which a characteristic of components of a system design has to be multiplied in order to reach equality of a characteristic of this design and different designs regarded as a standard. Equivalent of different system designs with respect to a reliability characteristic such as mean time to failure (MTTF) or survival function in case of no repairs is needed. Råde (1989) has introduced the concept of reliability equivalence. Råde (1990, 1991, 1993a, 1993b) has applied such concept to some simple series and parallel systems of two independent and identical components. Råde (1993a and 1993b) has used three different methods to improve the reliability of a system. In these methods it is assumed that the reliability of a system can be improved by: (i) improving the quality of one or several components by decreasing their failure rates; (ii) adding a hot component to the system; (iii) adding a cold redundant component to the system. Sarhan (2000) has introduced more generals improving methods of a system. In such methods, the reliability of a system can be improved by: (i) improving the quality of some components by reducing their failure rates by a factor ρ, $0< \rho <1$; (ii) assuming hot duplications of some components; (iii) assuming cold duplications of some components; (iv) assuming cold redundant standby components connected with some components (one for each) by random switches. Råde (1993a, 1993b) and Sarhan (2000) have used the survival function as the performance measure of the reliability system. Råde (1993a, 1993b) has calculated the reliability equivalence factors of series and parallel systems which consist of independent and identical components with constant failure rate. Sarhan (2000) has applied the concept of reliability equivalence on a series system consists of n independent but non-identical components. He assumed that the lifetime of each component follows the exponential distribution. Sarhan *et al.* (2004) have studied the equivalence of different designs of a four independent and identical components series-parallel system. They have tried to deduce the reliability equivalence factors of a series-parallel system. In obtaining these factors, the reliability function and mean time to failure (MTTF) of the system are used as performance measures to compare different system designs of original system and others improved systems. Sarhan (2005) obtained the reliability equivalence factors of a parallel system with n independent and identical components using the general method which is established in (2000). However, Sarhan (2000) obtained the MTTF in the case of independent and non-identical components series systems and some special cases studied in parallel system when the failure rates of the system's components are constants. Sarhan and Mustafa (2006) introduced different vectors of reliability equivalence factors of a series system consisting of n independent and non-identical components. They assumed that the failure rates of the system components are follow the exponential distribution and used the reliability function and the MTTF as performances to derive the reliability equivalence of the system. Their work generalized the results presented in Sarhan (2000). Sarhan *et al.* (2008) discussed the equivalence factors of m modules (subsystems) connected in series where each module i consisting of n_i components in parallel.

They assumed that the system components are independent and identical exponentially distributed. El-Damcese and Khlifa (2008) have considered the equivalence factors of m modules connected in parallel with subsystems i consisting of n_i components in series. They assumed that the system components are independent and identical Weibull distribution. Sarhan (2009) has studied the equivalence of different designs of a general series-parallel system based on the system reliability function and the MTTF. He assumed that the system components lives are independent and follow the exponential distributions. Generally speaking, to obtain the MTTF the first moment of order statistics is used, see, e.g., Arnold *et al.* (1992) and Asadi (2006). That is, compute $\mu_{n:n}=E(X_{n:n})$ for parallel system and $\mu_{1:n}=E(X_{1:n})$ for the series system. The *rth* moment of order statistics arising from independent non-identically distributed random variables have been studied by many authors see, e.g., Barakat and Abdelkader (2000, 2004), Abdelkader (2004 a and 2004 b), Abdelkader and Abotahoun (2006) and Abdelkader (2010, 2011).

Since the time to failure of an item is a random variable, Rausand and Høyland (2004) are used several measures of the center of a life distribution. The mean, the median and the mode are used to measure the time to failure of an item. Therefore, a higher moment such as the variance can also be used, the variance is one of the most commonly used measure for analyzing error. In this paper we shall use the survival function and the *rth* moment time to failure (MOTTF) to calculate the reliability equivalence factors for series and parallel systems, consisting of n independent and non-identical components. That is, we compute the *rth* moment time to failure, $\mu^{(r)}_{n:n}=E(X^{(r)}_{n:n})$, for the parallel system and, $\mu^{(r)}_{1:n}=E(X^{(r)}_{1:n})$, for the series system. These components are assumed to be follow the Weibull distribution in the case of independent non-identically distributed. The results presented here generalize those given in the literature.

The rest of the paper is organized as follows. The reliability functions and MOTTF for the parallel and series systems are introduced in Section 2. In Section 3, we derived the reliability functions and MOTTF for the parallel and series systems when the system components lives follow the Weibull distribution. Two different methods for improving the system designs are introduced in Section 4. In Section 5, the reliability equivalence factors and γ-fractiles of the original and improved designs are presented. Concluding remarks are drawn in Section 6.

2. The reliability functions and MOTTF

Assume a system composed of n independent but not identical components. Each component has a life time T_i, (i=1, 2,…, n). The reliability function of the system which consists of n components connected in parallel and series are defined, respectively, as

$$R(t)=1-\prod_{i=1}^{n}\left(1-R_i(t)\right), \qquad (1)$$

and

$$R(t)=\prod_{i=1}^{n}R_i(t), \qquad (2)$$

where R_i (t) is the reliability function of each component $i,(i=1,2,\ldots,n)$. The *MOTTF* is defined by

$$MOTTF=r\int_o^\infty t^{r-1}R(t)dt. \quad (3)$$

Let $MOTTF_p$ and $MOTTF_s$ be the moment time to failure for the parallel and series systems. They are, respectively, defined by

$$MOTTF_p=r\int_o^\infty t^{r-1}\left(1-\prod_{i=1}^{n}\left[1-R_i(t)\right]\right)dt, \quad (4)$$

and

$$MOTTF_s=r\int_o^\infty t^{r-1}\prod_{i=1}^{n}R_i(t)\,dt. \qquad (5)$$

3. The Weibull distribution

Let $T_i, i=1,2,\ldots,n$ be random variables having Weibull distribution with reliability functions

$$R_i(t)=e^{-a_i t^\beta}, t\ge 0,\ a_i,\beta>0. \quad (6)$$

The Weibull distribution generates a family of distributions as β changes its values. For instance, when β = 1 and β = 2, the Weibull distribution reduces to exponential and Rayleigh distribution with parameters, α_i, i = 1,2,…, respectively.

The Weibull distribution has a broad variety of monotone increasing failure rates when $\beta\ge 1$. That is, the failure rates, $r_i(t)$, , are functions of time

$$r_i(t)=\beta\,\alpha_i\,t^{\beta-1} \qquad (7).$$

The moment time to failure for the parallel and series systems will be derived in the following Theorem.

Theorem 1. The $MOTTF_p$ and $MOTTF_s$ for the Weibull distribution are given, respectively, by

$$MOTTF_p=\Gamma(\frac{r}{\beta}+1)\sum_{j=1}^{n}(-1)^{j+1}D_j, \quad (8)$$

and

$$MOTTF_s=\Gamma(\frac{r}{\beta}+1)D_n, \quad (9)$$

Where,

$$D_j = \sum_{1\le i_1<i_2\ldots} \cdots \sum_{<i_j\le n} \frac{1}{\left(\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_j}\right)^{\frac{r}{\beta}}},$$

$(i_1, i_2,\ldots, i_n)$ are the permutations of $(1,2,\ldots,n)$ for which $1\le i_1 < i_2 < \ldots < i_j \le n$ and $\Gamma(.)$ is the gamma function.

Proof. It is easy to see that

$$MOTTF_p = r\int_o^\infty t^{r-1}\left(1-\prod_{i=1}^{n}(1-e^{-\alpha_i t^\beta})\right)dt$$

$$= r\int_o^\infty t^{r-1}\Big(\sum_{i=1}^{n} e^{-\alpha_i t^\beta} - \sum_{1\le i_1\le}\sum_{i_2\le n} e^{-(\alpha_{i_1}+\alpha_{i_2})t^\beta} +$$

$$\sum_{1\le i_1<}\sum_{i_2<}\sum_{i_3\le n} e^{-(\alpha_{i_1}+\alpha_{i_2}+\alpha_{i_3})t^\beta} + \ldots + (-1)^{n+1} e^{-\sum_{i=1}^{n}\alpha_i t^\beta}\Big)dt.$$

Upon using

$$\int_o^\infty t^{r-1} e^{-\theta\, t^\beta} dt = \frac{\Gamma(\frac{r}{\beta})}{\beta\theta^{\frac{r}{\beta}}}, \quad \forall\,\theta>0, \quad (10)$$

we get Eq.(8). The proof of Eq.(9) follows from the definition

$$MOTTF_s = r\int_o^\infty t^{r-1}\prod_{i=1}^{n} e^{-\alpha_i t^\beta} dt = r\int_o^\infty t^{r-1} e^{-\sum_{i=1}^{n}\alpha_i t^\beta} dt,$$

and using Eq.(10).

Corollary 1. The $MOTTF_p$ and $MOTTF_s$ for the exponential distribution are given, respectively, by

$$MOTTF_p = \Gamma(r+1)\sum_{j=1}^{n}(-1)^{j+1} D_j, \qquad (11)$$

and

$$MOTTF_s = \Gamma(r+1) D_n, \qquad (12)$$

Where,

$$D_j = \sum_{1\le i_1<i_2\ldots} \cdots \sum_{<i_j\le n} \frac{1}{\left(\alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_j}\right)^{r}}.$$

Proof. Setting β=1 in Theorem 1, we get the results.

4. Designs of improved the series and Parallel systems

In this section, we shall introduce two different methods to improve the system designs. The first is the reduction method and the second is called redundancy method which is composed of hot and cold duplication methods. In reduction method, it is assumed that the failure rates of some of the system components are reduced by a factor ρ, 0<ρ<1. In the hot duplication method, it is assumed that some of the system components are duplicated in parallel while some of the system components are duplicated in parallel by a perfect switch in the cold duplication method. Therefore, one can make equivalence between the reduction method and the duplication method based on some reliability measures. In other words, the design of the system that is improved according to the reduction method should be equivalent to the design of the system according to one of the redundancy method. The comparison of the designs produce the so-called reliability equivalence factors.

4.1 The reduction method for series and Parallel systems

In this method, it is assumed that the system can be improved by improving l, $1\le l\le n$ of its components. That is, the failure rates of l components are reduced from α_i to ρ α_i, 0<ρ<1. Let $R_{s,\rho}^{(l)}(t)$ and $R_{p,\rho}^{(l)}(t)$ denote the reliability function of the series and parallel systems improved by reducing the failure rates of l of its components by the factor ρ.

Theorem 2. The moments of the series and parallel systems under the reduction method, denoted by $MOTTF_{s,\rho}^{(l)}$ and $MOTTF_{p,\rho}^{(l)}$, are given, respectively, by

$$MOTTF_{s,\rho}^{(l)} = \frac{\Gamma(\frac{r}{\beta}+1)}{\left(\rho\sum_{i=1}^{l}\alpha_i+\sum_{i=l+1}^{n}\alpha_i\right)^{\frac{r}{\beta}}}, \qquad (13)$$

and

$$MOTTF_{p,\rho}^{(l)} = \Gamma(\frac{r}{\beta}+1)\Big(\sum_{j=1}^{l}(-1)^{j+1} Z_j(\rho) +$$

$$\sum_{k=l+1}^{n}(-1)^{k+1} Z_k - \sum_{j=1}^{l}\sum_{k=l+1}^{n}(-1)^{j+k+2} Y_{jk}(\rho)\Big), \quad (14)$$

Where,

$$Z_j(\rho) = \sum_{1\le i_1<i_2\ldots} \cdots \sum_{<i_j\le l} \frac{1}{\left(\rho\sum_{j=1}^{l}\alpha_{i_j}\right)^{\frac{r}{\beta}}},$$

$$Z_k = \sum_{l+1\le i_1<i_2\ldots} \cdots \sum_{<i_k\le n} \frac{1}{\left(\sum_{k=l+1}^{n}\alpha_{i_k}\right)^{\frac{r}{\beta}}},$$

$$Y_{jk}(\rho) = \sum_{\substack{1\le i_1<i_2<\ldots \\ l+1\le i_1<i_2\ldots}} \cdots \sum_{\substack{<i_j\le l \\ <i_k\le n}} \frac{1}{\left(\rho\sum_{j=1}^{l}\alpha_{i_j}+\sum_{k=l+1}^{n}\alpha_{i_k}\right)^{\frac{r}{\beta}}},$$

The subscript of the last summation means we have two summations. One runs for j to l and the other runs for k to n.

Proof. The reliability function of the series system, $R_{s,\rho}^{(l)}(t)$ is given by

$$R_{s,\rho}^{(l)}(t) = \prod_{i=1}^{l} e^{-\rho\alpha_i t^\beta} \times \prod_{i=l+1}^{n} e^{-\alpha_i t^\beta}$$

$$= e^{-(\sum_{i=1}^{l} \rho\alpha_i + \sum_{i=l+1}^{n} \alpha_i) t^\beta} \qquad (15)$$

The reliability function of the parallel system, $R_{p,\rho}^{(l)}(t)$, is given by

$$R_{p,\rho}^{(l)}(t) = 1 - \prod_{i=1}^{l} (1 - e^{-\rho\alpha_i t^\beta}) \times \prod_{i=l+1}^{n} (1 - e^{-\alpha_i t^\beta})$$

$$= \sum_{j=1}^{l} (-1)^{j+1} Z_{i_j}(\rho t) + \sum_{k=l+1}^{n} (-1)^{k+1} Z_{i_k}(t) - \sum_{j=1}^{l} \sum_{k=l+1}^{n} (-1)^{j+k+2} Z_{i_j}(\rho t) Z_{jk}(t), \quad (16)$$

where, $Z_{i_j}(\rho t) = \sum_{1 \le i_1 < i_2 \ldots < i_j \le l} \ldots \sum e^{-\rho \sum_{j=1}^{l} \alpha_{i_j} t^\beta}$,

$$Z_{i_k}(t) = \sum_{l+1 \le i_1 < i_2 \ldots < i_k \le n} \ldots \sum e^{-\sum_{j=1}^{l} \alpha_{i_j} t^\beta}.$$

From the definitions of the moments which are given by Eqs. (4) and (5) and using the integral in Eq. (10), we get the results.

4.2 The hot duplication method

In this case, it is assumed that the system can be improved by improving m of its components, where $1 \le m \le n$.

Lemma 1. The reliability function of the parallel and series systems, denoted by $R_{p,H}^{(m)}(t)$ and $R_{s,H}^{(m)}(t)$, under the hot duplication method are given by

$$R_{p,H}^{(m)}(t) = \sum_{j=1}^{q} (-1)^{j+1} I_j, \qquad (17)$$

$$R_{s,H}^{(m)}(t) = I_q, \qquad (18)$$

where, $I_j = \sum_{1 \le i_1 < i_2 \ldots < i_j \le q} \ldots \sum e^{-\sum_{v=1}^{l} \alpha_{i_v} t^\beta}$,

and q=n+m.

Proof. The reliability function of the system, $R_{p,H}^{(m)}(t)$, is given by

$$R_{p,H}^{(m)}(t) = 1 - \prod_{i=1}^{q} (1 - e^{-\alpha_i t^\beta})$$

$$= \sum_{i=1}^{q} e^{-\alpha_i t^\beta} - \sum_{1 \le i_1 \le i_2 \le q} \sum e^{-(\alpha_{i_1} + \alpha_{i_2}) t^\beta} +$$

$$\sum_{1 \le i_1 < i_2 < i_3 \le q} \sum \sum e^{-(\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3}) t^\beta} + \ldots + (-1)^{q+1} e^{-\sum_{i=1}^{q} \alpha_i t^\beta}$$

We can rewrite the last equation in the form of Eq.(17). The proof of Eq.(18) follows from the definition

$$R_{s,H}^{(m)}(t) = \prod_{i=1}^{q} e^{-\alpha_i t^\beta} = e^{-\sum_{i=1}^{q} \alpha_i t^\beta} = I_q,$$

and hence the proof.

Theorem 3. The moment time to failure for the parallel and series systems, $MOTTF_{p,H}^{(m)}$ and $MOTTF_{s,H}^{(m)}$, are given by

$$MOTTF_{p,H}^{(m)} = \Gamma(\frac{r}{\beta} + 1) \sum_{j=1}^{q} (-1)^{j+1} D_j, \quad (19)$$

$$MOTTF_{s,H}^{(m)} = \Gamma(\frac{r}{\beta} + 1) D_q, \quad (20)$$

where,

$$D_j = \sum_{1 \le i_1 < i_2 \ldots < i_j \le q} \ldots \sum \frac{1}{(\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_j})^{\frac{r}{\beta}}}.$$

Proof. By replacing n by q in Theorem 1, we get the results.

4.3 The cold duplication method

Under the cold duplication method, the system will be improved by improving m of its components, $1 \le m \le n$. Let $R_{s,C}^{(m)}(t)$ and $R_{p,C}^{(m)}(t)$ be the reliability functions of the series and parallel systems. Then

$$R_{s,C}^{(m)}(t) = \prod_{i=1}^{m} R_{1,C}(t) \prod_{i=m+1}^{n} R_1(t), \qquad (21)$$

and

$$R_{p,C}^{(m)}(t) = 1 - \prod_{i=1}^{m} (1 - R_{1,C}(t)) \prod_{i=m+1}^{n} (1 - R_1(t)), \qquad (22)$$

where $R_{1,C}(t)$ denotes the reliability of a system's components after it was improved according to cold duplication method. So, we can write

$$R_{1,C}(t) = (1 + \alpha_i t^\beta) e^{-\alpha_i t^\beta}, \qquad 1 \le i \le m, \quad (23)$$

see Billinton and Allan (1983).

Theorem 4. The moments of the series and parallel systems, written as $MOTTF_{s,C}^{(m)}$ and $MOTTF_{p,C}^{(m)}$, in cold duplication method are given, respectively, by

$$MOTTF_{s,C}^{(m)}=\sum_{a_1=0}^{1}\cdots\sum_{a_m=0}^{1}\frac{r\prod_{i=1}^{m}\alpha_i^{a_i}\Gamma(\frac{r+\sum_{i=1}^{m}a_i}{\beta})}{\beta(\sum_{i=1}^{n}\alpha_i)^{(\frac{r+\sum_{i=1}^{m}a_i}{\beta})}}, \qquad (24)$$

and

$$MOTTF_{p,C}^{(m)}=\frac{r}{\beta}(\sum_{j=1}^{m}(-1)^{j+1}C_1(i_j)T_j+$$

$$\sum_{k=m+1}^{n}(-1)^{k+1}C_2(i_k)T_k-$$

$$\sum_{j=1}^{m}\sum_{k=m+1}^{n}(-1)^{j+k+2}C_1(i_j)C_2(i_k)T_{jk}), \quad (25)$$

Where,

$$C_1(i_j)=\sum_{1\le i_1<i_2\ldots}\cdots\sum_{<i_j\le m}\sum_{a_1=0}^{1}\cdots\sum_{a_m=0}^{1}\prod_{i=1}^{m}\alpha_{i_j}^{a_j},$$

$$C_2(i_k)=\sum_{m+1\le i_1<i_2\ldots<i_k\le n}\cdots\sum,$$

$$T_j=\frac{\Gamma(r+\sum_{i=1}^{m}a_i)}{(\sum_{j=1}^{m}\alpha_{i_j})^{(r+\Sigma_{i=1}^{m}a_i)}},$$

$$T_k=\frac{\Gamma(r)}{(\sum_{k=m+1}^{n}\alpha_{i_k})^{r}},$$

$$T_{jk}=\frac{\Gamma(r+\sum_{i=1}^{m}a_i)}{(\sum_{j=1}^{m}\alpha_{i_j}+\sum_{k=m+1}^{n}\alpha_{i_k})^{(r+\Sigma_{i=1}^{m}a_i)}}.$$

Proof. From Eq.(21), we can write

$$R_{s,C}^{(m)}(t)=\prod_{i=1}^{m}(1+\alpha_i t^{\beta})e^{-\alpha_i t^{\beta}}\prod_{i=m+1}^{n}e^{-\alpha_i t^{\beta}},$$

$$=e^{-\sum_{i=m+1}^{n}\alpha_i t^{\beta}}\prod_{i=1}^{m}\sum_{a=0}^{1}(\alpha_i t^{\beta})^{a}e^{-\alpha_i t^{\beta}}$$

$$=e^{-\sum_{i=m+1}^{n}\alpha_i t^{\beta}}T_m(t),$$

where,

$$T_m(t)=\sum_{a_1=0}^{1}\cdots\sum_{a_m=0}^{1}\prod_{i=1}^{m}\alpha_i^{a_i}t^{\beta\sum_{i=1}^{m}a_i}e^{-\sum_{i=1}^{m}\alpha_i t^{\beta}}.$$

Therefore,

$$MOTTF_{s,C}^{(m)}=r\int_{o}^{\infty}t^{r-1}R_{s,C}^{(m)}(t)dt.$$

Upon using the integral of Eq. (10), we get Eq. (24).

The reliability function of the parallel system can be written in the form

$$R_{p,C}^{(m)}(t)=\sum_{j=1}^{m}(-1)^{j+1}T_{i_j}(t)+\sum_{k=m+1}^{n}(-1)^{k+1}T_{i_k}(t)-$$

$$\sum_{j=1}^{m}\sum_{k=m+1}^{n}(-1)^{j+k+2}T_{i_j}(t)T_{jk}(t). \quad (26)$$

Since we can write

$$\prod_{i=m+1}^{n}(1-e^{-\alpha_i t^{\beta}})=1-\sum_{k=m+1}^{n}(-1)^{k+1}T_{i_k}(t),$$

$$T_{i_k}(t)=\sum_{m+1\le i_1<i_2\ldots}\cdots\sum_{<i_k\le n}e^{-\sum_{k=m+1}^{n}\alpha_{i_k}t^{\beta}},$$

$$\prod_{i=1}^{m}\left(1-(1+\alpha_i t^{\beta})e^{-\alpha_i t^{\beta}}\right)=1-\sum_{j=1}^{m}(-1)^{j+1}T_{i_j}(t),$$

$$T_{i_j}(t)=\sum_{1\le i_1<i_2\ldots}\cdots\sum_{<i_j\le m}\sum_{a_1=0}^{1}\cdots\sum_{a_m=0}^{1}\prod_{i=1}^{m}\alpha_{i_j}^{a_j}t^{\beta\sum_{j=1}^{m}a_j}e^{-\sum_{j=1}^{m}\alpha_{i_j}t^{\beta}}.$$

Therefore, Eq.(26) can be obtained. Hence,

$$MOTTF_{p,C}^{(m)}=r\int_{o}^{\infty}t^{r-1}R_{p,C}^{(m)}(t)dt,$$

Upon using the integral of Eq. (10), we get Eq. (25) and hence the proof.

5. The reliability equivalence factors and γ -Fractiles

In this section, the survival and the moments reliability equivalence factors for the series and parallel system will be introduced. The γ -Fractiles of the original and improved systems are also presented.

5.1 Hot-Cold series and parallel reliability equivalence factors

The survival reliability equivalence factors, written $\rho_F=\rho_{G,F}^{K}(\gamma)$ where G=s or p (for series and parallel systems); F=H(C) for hot(cold); $K=l$ or m components and $0<\gamma<1$, is defined as that factor ρ by which the failure rates of l of the system's components should be reduced in order to reach the reliability of the system which improved by improving m of the original system components according to hot(cold) duplication methods see, Sarhan (2009). The following set of non-linear equations give the solutions of ρ_F for the series and parallel systems:

$$R_{G,\rho}^{(l)}(t)=R_{G,F}^{(m)}(t)=\gamma.$$

Therefore, from Eqs. (15) - (18) we can write for the hot reliability equivalence factors (series and parallel systems) that

$$R_{s,\rho}^{(l)}(t)=e^{-(\rho\sum_{i=1}^{l}\alpha_i+\sum_{i=l+1}^{n}\alpha_i)t^{\beta}}$$

$$=R_{s,H}^{(m)}(t)=e^{-\sum_{i=1}^{n}\alpha_i t^{\beta}}=\gamma, \qquad (27)$$

$$R_{p,\rho}^{(l)}(t) = 1 - \prod_{i=1}^{l} \left(1 - e^{-\rho\alpha_i t^\beta}\right) \times \left(1 - e^{-\alpha_i t^\beta}\right)$$
$$= R_{s,H}^{(m)}(t) = \sum_{j=1}^{q} (-1)^{j+1} I_j = \gamma, \qquad (28)$$

where I_j is defined in Lemma 1.

Similar equations for the cold reliability equivalence factors (series and parallel systems) can be written using the Eqs. (15), (21) and (16), (22). That is,

$$R_{s,\rho}^{(l)}(t) = R_{s,C}^{(m)}(t) = \gamma, \qquad (29)$$

and

$$R_{p,\rho}^{(l)}(t) = R_{p,C}^{(m)}(t) = \gamma, \qquad (30)$$

The solutions of the system of Eqs. (27) - (30) have no closed form. So, a numerical technique can be used to get the solutions with respect to ρ_F .

On the other hand, the moment reliability equivalence factors can be used to reach the equality (or nearness) of the moment time to failure in the reduction method and the moment time to failure of the designs obtained from the hot(cold) duplication methods for the series and parallel systems, see Sarhan (2009). That is,

$$MOTTF_{s,\rho}^{(l)} = MOTTF_{s,H}^{(m)} = MOTTF_{s,C}^{(m)}, \qquad (31)$$

$$MOTTF_{p,\rho}^{(l)} = MOTTF_{p,H}^{(m)} = MOTTF_{p,C}^{(m)}. \qquad (32)$$

Therefore, using Eqs. (13), (20), (24) and (14), (19), (25) we get non-linear system of equations which can be solves numerically.

5.2 γ -Fractiles

Let $L(\beta,\gamma)$ be the γ -Fractiles of the original systems and $L_{G,F}^{(m)}(\beta,\gamma)$ be the γ -Fractiles of the improving systems obtained by improving *m* of the system's components according to G=s(p) for series and parallel systems and F=H(C) duplication methods.

The γ -Fractiles of the systems having reliability functions $R_G(t)$ and $L_G(\beta,\gamma)$ can be obtained by solving the equations

$$R\left(\frac{L_G(\beta,\gamma)}{\alpha}\right) = \gamma, \qquad (33)$$

and for the hot and cold duplication methods, we have to solve the following equations

$$R_{G,F}^{(m)}\left(\frac{L_{G,F}^{(m)}(\beta,\gamma)}{\alpha}\right) = \gamma, \qquad (34)$$

with respect to L. The above equations have no closed form solutions in L. So, the numerical techniques are the best solutions in these cases.

6. Concluding Remarks

The case of independent non-identical arises in many situations, we enumerate two. The first, we are not guarantee that all components of the system are work with the same efficiency even they are produced from the same factory. The second, we may assume that the components are came from different factories with different efficiency.

The results of this paper extend those available in the literature see, e.g., Sarhan (2000, 2002, 2005, 2009), Montaser and Sarhan (2008) and Sarhan *et al.* (2004, 2008). The system reliability function and the system moment time to failure (MOTTF) in the case of independent non-identical components are used to study the equivalence of different designs for the series and parallel systems. Moreover, the mentioned papers in the references assumed that the failure rates of the system components are follow the exponential distribution (constant failure rates) and computed the mean time to failure. While this paper presents the MOTTF and assumes the failure rates of the system components are functions of time and follow the Weibull distribution.

It is obvious from the foregoing Theorems to see that the following remarks: (i) from Eqs. (9) and (13), the $MOTTF_{s,\rho}^{(l)}$ is greater than or equal $MOTTF_s$ after reducing the failure rates of l of the components by the factor ρ Since the denominator of Eq. (9) is greater than the denominator of Eq. (13); (ii) similarly, the $MOTTF_{p,\rho}^{(l)}$ is greater than $MOTTF_p$, which are given by Eqs. (8) and (14), for the same reason; (iii) also, it is easy to deduce that $MOTTF_{s,C}^{(m)} > MOTTF_{s,H}^{(m)}$ and $MOTTF_{p,c}^{(m)} > MOTTF_{p,H}^{(m)}$. That is, the moment time to failure in the cold duplication method is greater than the moment time to failure in the hot duplication method. In reliability equivalence factors including the survival and the moments, most of the results established in the literature showed that

$$R_s(t) < R_{s,H}^{(m)}(t) < R_{s,C}^{(m)}(t).$$

and

$$R_p(t) < R_{p,H}^{(m)}(t) < R_{p,C}^{(m)}(t).$$

$$MOTTF_{s,\rho}^{(l)} < MOTTF_{s,H}^{(m)} < MOTTF_{s,C}^{(m)},$$

$$MOTTF_{p,\rho}^{(l)} < MOTTF_{p,H}^{(m)} < MOTTF_{p,C}^{(m)},$$

apart from the components are independent identical or non-identical see, e.g., Sarhan (2000, 2002, 2005, 2009), Sarhan and Mustafa (2006), Sarhan *et al.* (2004, 2008), Xia and Zhang (2007), El-Damcese and Khalifa (2008) and Montaser and Sarhan (2008).

Moreover, the results in the mentioned references showed that the amount of increasing of the γ -fractile in cold duplication method is greater than the γ -fractile obtained by using the hot duplication method.

In the future, we may study the reliability equivalence factors for other family of distributions in the case of independent non-identical components along with high quality software programs to perform the associated calculations.

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