Calculation of Compressible Flow Past a Joukowski Aerofoil Using Direct Boundary Element Method with Constant Element Approach

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Abstract: In this paper, a steady, inviscid compressible flow past a Joukowski aerofoil has been calculated using direct boundary element method (DBEM) with constant boundary elements the velocity distribution for the flow over the surface of the Joukowski aerofoil which have been compared with the analytical results.
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Keywords: Direct boundary element method, Inviscid Compressible flow, Joukowski aerofoil.

1. Introduction

In recent past, the well-known computational methods such as finite difference method (FDM), finite element method (FEM), and boundary element method (BEM) etc have been applied for the flow field calculations around bodies. Out of these methods, BEM is a modern numerical technique in which only the boundary of the body under consideration is discretized in to different type of elements. BEM is well-suited to problems where domain is exterior to the boundary, as in the case of flow past bodies. The most important features of BEM is the much smaller system of equations and considerable reduction in data, which are essential to run a computer program efficiently. That is why; BEM is more accurate, efficient and economical than other domain methods. The BEM can be classified into two categories i.e. direct and indirect. (see Brebbia and Walker, 1978 & 1980, Ramsey, 1942, Milne-Thomson, 1968, & Kellogge, 1929). The direct and indirect methods have been used in the past for flow field calculations around bodies (Morino 1975, Hess & Smith, 1967, Kohr, 2000, Luminita, 2008, Muhammad, 2009; Mushtaq, 2008, 2009, 2010, 2011& 2012). Most of the work on fluid flow calculations using boundary element methods has been done in the field of incompressible flow. Very few attempts have been made on flow field calculations using boundary element methods in the field of compressible flow. In this paper, the DBEM has been used for the solution of inviscid compressible flow around a Joukowski aerofoil.

2. Mathematical Formulation of Steady and Inviscid Compressible Flow

We know that equation of motion for two – dimensional, steady, irrotational, and isentropic flow (Mushtaq, 2010, 2011 & 2012, Shah, 2011) is

$$(1 - Ma^2)\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} = 0 \qquad (1)$$

where Ma is the Mach number and Φ is the total velocity potential of the flow. Here X and Y are the space coordinates.

Using the dimensionless variables, $x = X$, $y = \beta\, Y$, where $\beta = \sqrt{1 - Ma^2}$, equation (1) becomes

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

or $\nabla^2 \Phi = 0$ (2)

which is Laplace's equation.

3. Steady and Inviscid Compressible Flow Past a Joukowski Aerofoil

Consider the flow past a Joukowski aerofoil and let the onset flow be the uniform stream with velocity U in the positive direction of the x – axis as shown in figure (1).

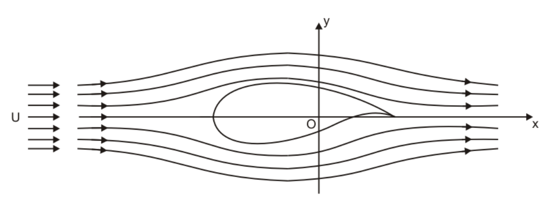


Figure 1: Flow past a Joukowski aerofoil.

Exact Velocity

The magnitude of the exact velocity distribution over the boundary of a Joukowski aerofoil is given by [Chow, 1979; Mushtaq, 2011 & 2012]

$$V = U\left|\frac{1-\dfrac{r^2}{(z-z_1)^2}+\dfrac{2ic}{(z-z_1)}}{1-\left(\dfrac{a}{z}\right)^2}\right|$$

where r = radius of the cylinder,
a = Joukowski transformation constant
$z = x+iy, \quad z_1 = b+ic,$
$b = a-\sqrt{r^2-c^2}$

In Cartesian coordinates the exact velocity becomes

$$V = U\,\frac{\left[\left[\{(x-b)^2+(y-c)^2\}^2 - r^2\{(x-b)^2-(y-c)^2\} + 2c(y-c)\{(x-b)^2+(y-c)^2\}\right]^2 + \left[2c(x-b)\{(x-b)^2+(y-c)^2\} + 2r^2(x-b)(y-c)\right]^2\right]^{1/2}}{[(x-b)^2+(y-c)^2]^2} \times \frac{\sqrt{[(x^2+y^2)^2-a^2(x^2-y^2)]^2+4a^4x^2y^2}}{(x^2-y^2-a^2)^2+4x^2y^2}$$

Boundary Conditions

Now the condition to be satisfied on the boundary of a Joukowski aerofoil is

$$\frac{\partial \Phi}{\partial n} = 0 \tag{3}$$

where Φ is the total velocity potential.

Now the total velocity potential Φ is the sum of the perturbation velocity potential $\phi_{j.a}$ where the subscript $j.a$ stands for Joukowski aerofoil and the velocity potential of the uniform stream $\phi_{u.s}$. (Mushtaq, 2010, 2011, & 2012).

i.e. $$\Phi = \phi_{u.s} + \phi_{j.a} \tag{4}$$

or $$\frac{\partial \Phi}{\partial n} = \frac{\partial \phi_{u.s}}{\partial n} + \frac{\partial \phi_{j.a}}{\partial n} \tag{5}$$

From equations (3) and (5), we get

$$\frac{\partial \phi_{j.a}}{\partial n} + \frac{\partial \phi_{u.s}}{\partial n} = 0$$

or $$\frac{\partial \phi_{j.a}}{\partial n} = -\frac{\partial \phi_{u.s}}{\partial n} \tag{6}$$

But the velocity potential of the uniform stream is given by [see Milne – Thomson, 1968 & Shah, 2008 & Mushtaq,2008,2009, 2010, 2011, &2012]

$$\phi_{u.s} = -Ux \tag{7}$$

$$= -U\frac{\partial x}{\partial n}$$

$$= -U(\hat{n}\cdot\hat{i}) \tag{8}$$

Thus from equations (6) and (8), we get

$$\frac{\partial u_{j.a}}{\partial n} = U(\hat{n}\cdot\hat{i}) \tag{9}$$

Now from the figure (2)

$$\vec{A} = (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j}$$

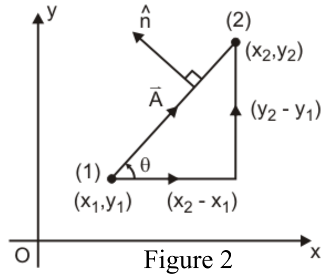


Figure 2

Therefore the unit vector in the direction of the vector $\vec{A}$ is given by

$$\hat{A} = \frac{(x_2-x_1)\hat{i} + (y_2-y_1)\hat{j}}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

The outward unit normal vector $\hat{n}$ to the vector $\vec{A}$ is given by

$$\hat{n} = \frac{-(y_2-y_1)\hat{i} + (x_2-x_1)\hat{j}}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

Thus $$\hat{n}\cdot\hat{i} = \frac{(y_1-y_2)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \tag{10}$$

From equations (9) and (10), we get

$$\frac{\partial \phi_{j.a}}{\partial n} = U\frac{(y_1-y_2)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \tag{11}$$

Equation (11) is the boundary condition which must be satisfied over the boundary of a Joukowski aerofoil.

Equation of Direct Boundary Element Method

The equation of DBEM for two–dimensional flow [see Mushtaq, 2008, 2009, 2010, 2011 & 2012] is :

$$-c_i\,\phi_i+\frac{1}{2\pi}\int_{\Gamma-i}\phi\frac{\partial}{\partial n}\left[\log\left(\frac{1}{r}\right)\right]d\Gamma+\phi_\infty$$

$$=\frac{1}{2\pi}\int_{\Gamma}\log\left(\frac{1}{r}\right)\frac{\partial\phi}{\partial n}\,d\Gamma \qquad (12)$$

where $c_i = 0$ when i is exterior to Γ
$= 1$ when i is interior to Γ
$= \frac{1}{2}$ when i lies on Γ and Γ is smooth.

Matrix Formulation with Constant Element Approach

The equation (12) for the DBEM can be written in the discretized form as

$$-c_i\,\phi_i+\sum_{j=1}^{m}\left[\int_{\Gamma_j-i}\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log\frac{1}{r}\right)d\Gamma\right]\phi_j+\phi_\infty$$

$$=\sum_{j=1}^{m}\left[\int_{\Gamma_j}\left(\frac{1}{2\pi}\log\frac{1}{r}\right)d\Gamma\right]\frac{\partial\phi_j}{\partial n} \qquad (13)$$

The integrals in equation (13) on the elements can be calculated numerically except the element on which the fixed point 'i' is located. For this element the integrals are calculated analytically. Denoting the integrals on the L.H.S. of equation (13) by $\hat{H}_{ij}$ and that on the R.H.S. by G_{ij}, then

$$\text{where } \hat{H}_{ij}=\int_{\Gamma_j-i}\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log\frac{1}{r}\right)d\Gamma \qquad (14)$$

$$\text{and } G_{ij}=\int_{\Gamma_j}\frac{1}{2\pi}\log\left(\frac{1}{r}\right)d\Gamma \qquad (15)$$

For the case of that element on which the fixed point 'i' is lying, these integrals have been calculated [see Mushtaq, 2008, 2009,2011 & 2012].

Thus equation (13) can be written as

$$-c_i\,\phi_i+\sum_{j=1}^{m}\hat{H}_{ij}\,\phi_j+\phi_\infty=\sum_{j=1}^{m}G_{ij}\frac{\partial\phi_j}{\partial n} \qquad (16)$$

Defining

$$H_{ij}=\begin{cases}\hat{H}_{ij} & \text{when } i\neq j\\ \hat{H}_{ij}-c_i & \text{when } i=j\end{cases}$$

Equation (16) takes the form

$$\sum_{j=1}^{m}H_{ij}\,\phi_j+\phi_\infty=\sum_{j=1}^{m}G_{ij}\frac{\partial\phi_j}{\partial n}$$

which can be expressed in matrix form as

$$[H]\,\{\underline{U}\}=[G]\,\{\underline{Q}\} \qquad (17)$$

Since $\frac{\partial\phi}{\partial n}$ is specified at each node of the element, the values of the perturbation velocity potential ϕ are found at each node on the boundary via equation (17). The total potential Φ is then found from equation (4) which will then be used to calculate the velocity on the symmetric aerofoil.

The velocity midway between two nodes on the boundary can then be approximated by using the formula

$$\text{Velocity } \vec{V}=\frac{\Phi_{k+1}-\Phi_k}{\text{Length from node } k \text{ to } k+1} \qquad (18)$$

Process of Discretization

Now for the discretization of the boundary of the Joukowski aerofoil, the coordinates of the extreme points of the boundary elements can be generated within computer programme using Fortran language as follows:

Divide the boundary of the circular cylinder into m elements in the clockwise direction by using the formula.

$$\theta_k=[\,(m+3)-2k\,]\frac{\pi}{m},$$

$$k=1,2,\ldots\ldots,m \qquad (19)$$

Then the extreme points of these m elements of circular cylinder are found by

$$\xi_k=-b+r\cos\theta_k$$
$$\eta_k=c+r\sin\theta_k$$

Now by using Joukowski transformation

$$z=\zeta+\frac{a^2}{\zeta}$$

the extreme points of the Joukowski aerofoil are

$$x_k=\xi_k\left(1+\frac{a^2}{\xi_k^2+\eta_k^2}\right)$$

$$y_k=\eta_k\left(1-\frac{a^2}{\xi_k^2+\eta_k^2}\right)$$

where $k=1,2,\ldots\ldots,m$.

The coordinates of the middle node of each boundary element are given by

$$\left.\begin{aligned}x_m&=\frac{x_k+x_{k+1}}{2}\\ y_m&=\frac{y_k+y_{k+1}}{2}\end{aligned}\right\}$$

$$k,m=1,2,\ldots\ldots,n \qquad (20)$$

and therefore the boundary condition (11) in this case takes the form

$$\frac{\partial\phi_{j,a}}{\partial n}=U\frac{(y_1)_m-(y_2)_m}{\sqrt{[(x_2)_m-(x_1)_m]^2+[(y_2)_m-(y_1)_m]^2}} \qquad (21)$$

The following tables show the comparison of computed and analytical velocity distribution over the boundary of a Joukowski aerofoil for 8, 16, 32, and 64 boundary elements with constant element approach.

Table (1)

ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-13.22	2.60	13.47	.80735E+00	.80488E+00
2	-9.75	6.06	11.48	.19490E+01	.18878E+01
3	-4.85	6.06	7.76	.19482E+01	.18886E+01
4	-1.39	2.60	2.94	.80617E+00	.79986E+00
5	-1.39	-2.29	2.68	.80588E+00	.71925E+00
6	-4.85	-5.76	7.53	.19482E+01	.18086E+01
7	-9.75	-5.76	11.33	.19490E+01	.18078E+01
8	-13.22	-2.30	13.41	.80737E+00	.72490E+00

Table (2)

ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-14.38	1.56	14.46	.39531E+00	.42990E+00
2	-13.30	4.16	13.93	.11257E+01	.11506E+01
3	-11.31	6.15	12.87	.16847E+01	.17026E+01
4	-8.71	7.22	11.31	.19871E+01	.20020E+01
5	-5.89	7.22	9.32	.19868E+01	.20025E+01
6	-3.29	6.14	6.97	.16836E+01	.17033E+01
7	-1.30	4.15	4.35	.11227E+01	.11487E+01
8	-.22	1.55	1.57	.39395E+00	.41741E+00
9	-.22	-1.25	1.27	.39290E+00	.33444E+00
10	-1.30	-3.84	4.06	.11223E+01	.10686E+01
11	-3.29	-5.84	6.71	.16836E+01	.16233E+01
12	-5.89	-6.92	9.09	.19869E+01	.19225E+01
13	-8.71	-6.92	11.13	.19872E+01	.19220E+01
14	-11.31	-5.85	12.73	.16848E+01	.16226E+01
15	-13.30	-3.86	13.85	.11258E+01	.10706E+01
16	-14.38	-1.26	14.43	.39532E+00	.34992E+00

Table (3)

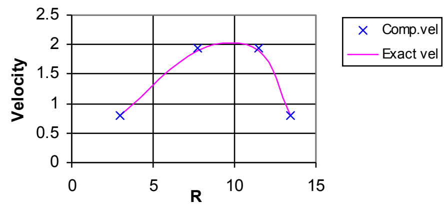
ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-14.69	.88	14.72	.19670E+00	.23588E+00
2	-14.41	2.31	14.59	.58254E+00	.62017E+00
3	-13.85	3.65	14.33	.94599E+00	.98225E+00
4	-13.04	4.86	13.92	.12731E+01	.13082E+01
5	-12.01	5.89	13.38	.15512E+01	.15856E+01
6	-10.80	6.70	12.71	.17697E+01	.18037E+01
7	-9.46	7.26	11.92	.19202E+01	.19541E+01
8	-8.03	7.54	11.01	.19968E+01	.20309E+01
9	-6.57	7.54	10.00	.19967E+01	.20312E+01
10	-5.14	7.25	8.89	.19197E+01	.19548E+01
11	-3.80	6.70	7.70	.17689E+01	.18045E+01
12	-2.59	5.89	6.43	.15499E+01	.15859E+01
13	-1.56	4.85	5.10	.12712E+01	.13074E+01
14	-.75	3.64	3.72	.94303E+00	.97909E+00
15	-.19	2.28	2.29	.57662E+00	.61237E+00
16	.10	.87	.87	.19624E+00	.21350E+00
17	.10	-.55	.56	.19311E+00	.12550E+00
18	-.19	-1.98	1.99	.57418E+00	.53113E+00
19	-.75	-3.34	3.42	.94261E+00	.89879E+00
20	-1.56	-4.55	4.81	.12711E+01	.12273E+01
21	-2.59	-5.59	6.16	.15499E+01	.15059E+01
22	-3.80	-6.40	7.44	.17689E+01	.17245E+01

23	-5.14	-6.95	8.65	.19197E+01	.18748E+01
24	-6.57	-7.24	9.78	.19967E+01	.19512E+01
25	-8.03	-7.24	10.81	.19969E+01	.19509E+01
26	-9.46	-6.96	11.74	.19202E+01	.18741E+01
27	-10.80	-6.40	12.56	.17697E+01	.17237E+01
28	-12.01	-5.59	13.25	.15512E+01	.15056E+01
29	-13.04	-4.56	13.82	.12731E+01	.12282E+01
30	-13.85	-3.35	14.25	.94600E+00	.90226E+00
31	-14.41	-2.01	14.55	.58255E+00	.54020E+00
32	-14.69	-.58	14.70	.19670E+00	.15591E+00

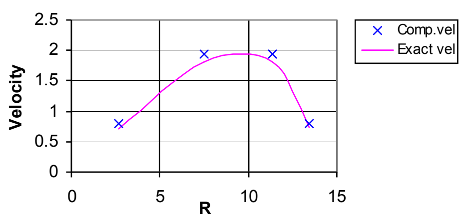
Table (4)

ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-14.77	.52	14.78	.98249E-01	.13805E+00
2	-14.70	1.25	14.76	.29374E+00	.33324E+00
3	-14.56	1.97	14.69	.48644E+00	.52562E+00
4	-14.35	2.67	14.59	.67447E+00	.71333E+00
5	-14.06	3.35	14.46	.85593E+00	.89459E+00
6	-13.72	4.00	14.29	.10292E+01	.10676E+01
7	-13.31	4.61	14.09	.11926E+01	.12308E+01
8	-12.84	5.17	13.85	.13444E+01	.13826E+01
9	-12.33	5.69	13.58	.14833E+01	.15214E+01
10	-11.76	6.16	13.27	.16079E+01	.16461E+01
11	-11.15	6.57	12.94	.17170E+01	.17552E+01
12	-10.50	6.91	12.57	.18096E+01	.18479E+01
13	-9.82	7.19	12.17	.18848E+01	.19232E+01
14	-9.12	7.41	11.75	.19417E+01	.19804E+01
15	-8.40	7.55	11.29	.19800E+01	.20188E+01
16	-7.67	7.62	10.81	.19992E+01	.20382E+01
17	-6.93	7.62	10.30	.19991E+01	.20384E+01
18	-6.20	7.55	9.77	.19798E+01	.20192E+01
19	-5.48	7.40	9.21	.19414E+01	.19810E+01
20	-4.78	7.19	8.63	.18842E+01	.19239E+01
21	-4.10	6.91	8.04	.18089E+01	.18487E+01
22	-3.45	6.56	7.42	.17161E+01	.17559E+01
23	-2.84	6.15	6.78	.16068E+01	.16466E+01
24	-2.28	5.69	6.13	.14819E+01	.15216E+01
25	-1.76	5.17	5.46	.13428E+01	.13821E+01
26	-1.29	4.60	4.78	.11906E+01	.12295E+01
27	-.88	3.99	4.08	.10268E+01	.10652E+01
28	-.54	3.34	3.38	.85289E+00	.89058E+00
29	-.26	2.66	2.67	.67049E+00	.70708E+00
30	-.04	1.95	1.95	.48092E+00	.51583E+00
31	.11	1.21	1.21	.28468E+00	.31686E+00
32	.21	.50	.54	.10199E+00	.10809E+00
33	.23	-.17	.29	.99324E-01	.49335E-01
34	.13	-.90	.90	.27830E+00	.23185E+00
35	-.04	-1.64	1.64	.47814E+00	.43395E+00
36	-.26	-2.35	2.37	.66932E+00	.62622E+00
37	-.54	-3.04	3.08	.85227E+00	.81015E+00
38	-.88	-3.69	3.79	.10264E+01	.98499E+00
39	-1.29	-4.30	4.49	.11903E+01	.11494E+01
40	-1.76	-4.87	5.17	.13426E+01	.13021E+01
41	-2.28	-5.39	5.85	.14819E+01	.14416E+01
42	-2.84	-5.85	6.51	.16068E+01	.15666E+01
43	-3.45	-6.26	7.15	.17161E+01	.16760E+01
44	-4.10	-6.61	7.78	.18089E+01	.17687E+01
45	-4.78	-6.89	8.39	.18842E+01	.18440E+01
46	-5.48	-7.10	8.97	.19414E+01	.19010E+01

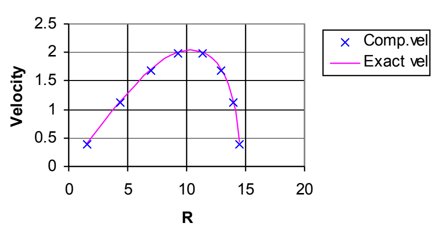
47	-6.20	-7.25	9.54	.19798E+01	.19392E+01
48	-6.93	-7.32	10.08	.19992E+01	.19584E+01
49	-7.67	-7.32	10.60	.19992E+01	.19582E+01
50	-8.40	-7.25	11.09	.19800E+01	.19388E+01
51	-9.12	-7.11	11.56	.19418E+01	.19004E+01
52	-9.82	-6.89	12.00	.18848E+01	.18432E+01
53	-10.50	-6.61	12.41	.18096E+01	.17679E+01
54	-11.15	-6.27	12.79	.17171E+01	.16752E+01
55	-11.76	-5.86	13.14	.16079E+01	.15660E+01
56	-12.33	-5.39	13.45	.14833E+01	.14414E+01
57	-12.84	-4.87	13.74	.13444E+01	.13026E+01
58	-13.31	-4.31	13.99	.11926E+01	.11508E+01
59	-13.72	-3.70	14.21	.10292E+01	.98766E+00
60	-14.06	-3.05	14.39	.85597E+00	.81461E+00
61	-14.35	-2.37	14.54	.67441E+00	.63336E+00
62	-14.56	-1.67	14.65	.48646E+00	.44564E+00
63	-14.70	-.95	14.73	.29374E+00	.25327E+00
64	-14.77	-.22	14.78	.98238E-01	.58082E-01



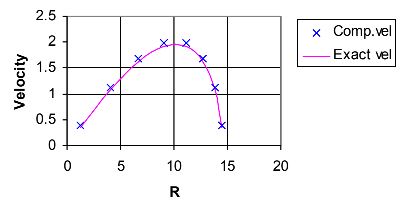
Graph 1: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using upper 4 values of 8 boundary elements with direct constant element approach for r =7.5, a=0.2, c=0.15, and Ma=0.7.



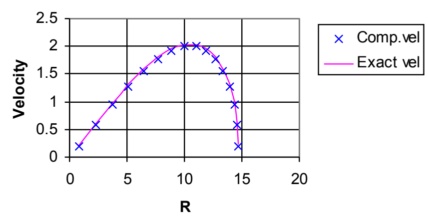
Graph 2: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using lower 4 values of 8 boundary elements with direct constant element approach for r=7.5, a=0.2, c=0.15, and Ma=0.7.



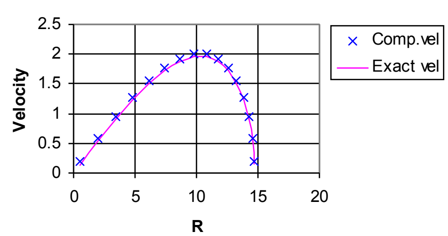
Graph 3: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using upper 8 values of 16 boundary elements with direct constant element approach for r=7.5, a=0.2, c=0.15, and Ma=0.7.



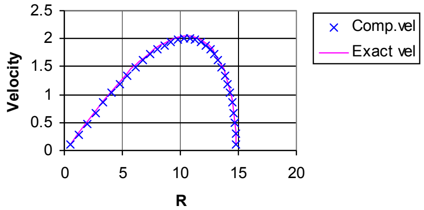
Graph 4: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using lower 8 values of 16 boundary elements with direct constant element approach for r=7.5, a=0.2, c=0.15, and M =0.7.



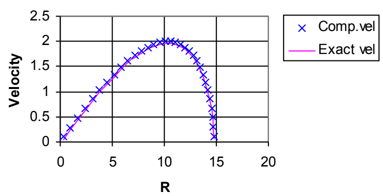
Graph 5: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using upper 16 values of 32 boundary elements with direct constant element approach for r = 7.5, a = 0.2, c = 0.15, and Ma = 0.7.



Graph 6: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using lower 16 values of 32 boundary elements with direct constant element approach for r = 7.5, a = 0.2, c = 0.15, and Ma = 0.7.



Graph 7: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using upper 32 values of 64 boundary elements with direct constant element approach for r = 7.5, a = 0.2, c = 0.15, and Ma = 0.7.



Graph 8: Comparison of computed and analytical velocity distributions over the boundary of a Joukowski aerofoil using lower 32 values of 64 boundary elements with direct constant element approach for r = 7.5, a = 0.2, c = 0.15, and Ma = 0.7.

4. Conclusion

We calculate the steady and inviscid compressible flow past a Joukowski aerofoil using DBEM with constant element approach. The calculated flow velocities obtained using this method is compared with the analytical solutions for flow over the boundary of a Joukowski aerofoil. It is found that from tables and graphs, the computed results obtained by this method are good in agreement with the analytical ones for the body under consideration and the accuracy of the result increases due to increase of number of boundary elements.

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