Direct Boundary Element Method for Calculation of Hyperbolic Flow Past a Sphere

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Abstract: In this paper, direct method is applied for calculating the hyperbolic flow past a sphere. The surface of the body is discretised into boundary elements on which the velocity distribution is found. The comparison of computed and exact results is also made.
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Introduction

Direct method is a numerical method which is in the form of a statement which gives the values of unknown variables at the field point under discussion in terms of a complete set of the entire boundary data. This method is used in different areas like solid and fracture mechanics, fluid dynamics and potential theory etc. [1]. The initial work for potential flow calculations was done by Hess and Smith ([2], [3]). Indirect method is popular due to its simplicity because the discretisation only takes place on the surface of the body. The direct method was applied for potential flow calculation in the past by Morino [4], Muhammad [8] and Mushtaq [9].

Calculation of Hyperbolic Flow Past a Sphere

Let a sphere of radius 'a' be taken as stationary and let U be the velocity of a uniform stream flowing in the positive direction of x – axis as shown in figure 1 [5].

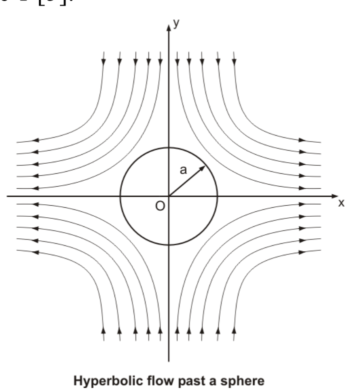

Figure (1)

The stream function in this case given by

$$\psi = \Omega x (y^2 + z^2) \left\{ 1 - \frac{5}{2}\left(\frac{a}{R_1}\right)^3 + \frac{3}{2}\left(\frac{a}{R_1}\right)^5 \right\} \qquad (1)$$

Since $y^2 + z^2 = r^2$

$$\psi = \frac{U x}{\sqrt{4x^2 + y^2 + z^2}} (y^2 + z^2) \left\{ 1 - \frac{5}{2}\left(\frac{a}{R_1}\right)^3 + \frac{3}{2}\left(\frac{a}{R_1}\right)^5 \right\}$$

$$= \frac{U x r^2}{\sqrt{4x^2 + r^2}} \left\{ 1 - \frac{5}{2}\left(\frac{a}{R_1}\right)^3 + \frac{3}{2}\left(\frac{a}{R_1}\right)^5 \right\} \qquad (2)$$

Since $v_x = -\dfrac{1}{r}\dfrac{\partial \psi}{\partial r}$

and $v_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial x}$

$$\frac{\partial \psi}{\partial r} = U \left[\left\{ \frac{2xr}{\sqrt{4x^2+r^2}} - \frac{x r^2 (2r)}{2(4x^2+r^2)^{3/2}} \right\} \left\{ 1 - \frac{5}{2}\left(\frac{a}{R_1}\right)^2 + \frac{3}{2}\left(\frac{a}{R_1}\right)^5 \right\} + \frac{x r^2}{\sqrt{4x^2+r^2}} \left\{ -\frac{15}{2}\left(\frac{a}{R_1}\right)^2 \frac{\partial}{\partial r}\left(\frac{a}{R_1}\right) + \frac{15}{2}\left(\frac{a}{R_1}\right)^4 \frac{\partial}{\partial r}\left(\frac{a}{R_1}\right) \right\} \right]$$

$$= U \left[\left\{ \frac{2xr}{\sqrt{4x^2+r^2}} - \frac{x r^3}{(4x^2+r^2)^{3/2}} \right\} \left\{ 1 - \frac{5}{2}\left(\frac{a}{R_1}\right)^2 + \frac{3}{2}\left(\frac{a}{R_1}\right)^5 \right\} + \frac{x r^2}{\sqrt{4x^2+r^2}} \left\{ -\frac{15}{2}\left(\frac{a}{R_1}\right)^2 \left(-\frac{a}{2R_1^3} 2r\right) + \frac{15}{2}\left(\frac{a}{R_1}\right)^4 \left(-\frac{a}{2R_1^3} 2r\right) \right\} \right]$$

$$= U \left[\left\{ \frac{2xr(4x^2+r^2) - x r^3}{(4x^2+r^2)^{3/2}} \right\} \left\{ 1 - \frac{5}{2}\left(\frac{a}{R_1}\right)^3 + \frac{3}{2}\left(\frac{a}{R_1}\right)^5 \right\} + \frac{x r^2}{\sqrt{4x^2+r^2}} \left\{ \frac{15ar}{2R_1^3}\left(\frac{a}{R_1}\right)^2 - \frac{15ar}{2R_1^3}\left(\frac{a}{R_1}\right)^4 \right\} \right]$$

$$= U\left[\frac{x\,r\,(8x^2+2r^2-r^2)}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{x\,r^2}{\sqrt{4x^2+r^2}}\;\frac{15\,r}{2}\left(\frac{a^3}{R_1^5}-\frac{a^5}{R_1^7}\right)\right]$$

$$= U\left[\frac{x\,r\,(8x^2+r^2)}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{x\,r^2}{\sqrt{4x^2+r^2}}\;\frac{15\,a^3 r}{2R_1^7}\left(R_1^2-a^2\right)\right]$$

$$= U\left[\frac{x\,r\,(8x^2+r^2)}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{15}{2}\,\frac{a^3 x\,r^3\left(R_1^2-a^2\right)}{R_1^7\sqrt{4x^2+r^2}}\right] \tag{3}$$

$$v_x = -\frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$= -\frac{U}{r}\left[\frac{x\,r\,(8x^2+r^2)}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{15}{2}\,\frac{a^3 x\,r^3\left(R_1^2-a^2\right)}{R_1^7\sqrt{4x^2+r^2}}\right]$$

$$= -U\left[\frac{x\,(8x^2+r^2)}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{15}{2}\,\frac{a^3 x\,r^2\left(R_1^2-a^2\right)}{R_1^7\sqrt{4x^2+r^2}}\right]$$

$$\frac{\partial\psi}{\partial x} = U$$

$$\left[\left\{\frac{r^2}{\sqrt{4x^2+r^2}}-\frac{x\,r^2(8x)}{2(4x^2+r^2)^{3/2}}\right\}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}\right.$$
$$\left.+\frac{x\,r^2}{\sqrt{4x^2+r^2}}\left\{-\frac{15}{2}\left(\frac{a}{R_1}\right)^2\frac{\partial}{\partial x}\left(\frac{a}{R_1}\right)+\frac{15}{2}\left(\frac{a}{R_1}\right)^4\frac{\partial}{\partial x}\left(\frac{a}{R_1}\right)\right\}\right]$$

$$= U\left[\left\{\frac{r^2(4x^2+r^2)-4x^2r^2}{(4x^2+r^2)^{3/2}}\right\}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}\right.$$
$$\left.+\frac{x\,r^2}{\sqrt{4x^2+r^2}}\left\{-\frac{15}{2}\left(\frac{a}{R_1}\right)^2\left(-\frac{a}{2R_1^3}2x\right)+\frac{15}{2}\left(\frac{a}{R_1}\right)^4\left(-\frac{a}{2R_1^3}2x\right)\right\}\right]$$

$$= U\left[\left\{\frac{r^2(4x^2+r^2-4x^2)}{(4x^2+r^2)^{3/2}}\right\}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}\right.$$
$$\left.+\frac{x\,r^2}{\sqrt{4x^2+r^2}}\left\{\frac{15}{2}\,\frac{a\,x}{R_1^3}\left(\frac{a}{R_1}\right)^2-\frac{15}{2}\,\frac{a\,x}{2R_1^3}\left(\frac{a}{R_1}\right)^4\right\}\right]$$

$$= U\left[\frac{r^4}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{x\,r^2}{\sqrt{4x^2+r^2}}\;\frac{15\,x}{2}\left(\frac{a^3}{R_1^5}-\frac{a^5}{R_1^7}\right)\right]$$

$$= U\left[\frac{r^4}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{x\,r^2}{\sqrt{4x^2+r^2}}\;\frac{15\,x\,a^3}{2R_1^7}\left(R_1^2-a^2\right)\right]$$

$$= U\left[\frac{r^4}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{15}{2}\,\frac{a^3x^2r^2\left(R_1^2-a^2\right)}{\sqrt{4x^2+r^2}}\right]$$

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial x}$$

$$= U\left[\frac{r^4}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{15}{2}\,\frac{a^3x^2r^2\left(R_1^2-a^2\right)}{\sqrt{4x^2+r^2}}\right]$$

$$= U\left[\frac{r^3}{(4x^2+r^2)^{3/2}}\left\{1-\frac{5}{2}\left(\frac{a}{R_1}\right)^3+\frac{3}{2}\left(\frac{a}{R_1}\right)^5\right\}+\frac{15}{2}\,\frac{a^3x^2r\left(R_1^2-a^2\right)}{R_1^7\sqrt{4x^2+r^2}}\right] \tag{4}$$

Equation of Direct Boundary Element Method

The equation of direct boundary element method for three – dimensional problems is given by (see [6, 7, 8, 9]).

$$c_i\,\phi_i = \phi_\infty - \frac{1}{4\pi}\iint_S \frac{1}{r}\frac{\partial\phi}{\partial n}\,dS + \frac{1}{4\pi}\iint_{S-i}\phi\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{5}$$

Discretization of Sphere:

The surface of the sphere is discretized into quadrilateral elements. The scheme of discretization is as shown in the figure (2).

The direct boundary element method is applied to calculate the hyperbolic flow solution around the sphere for which the analytical solution is available

Consider the surface of the sphere in one octant to be divided into three quadrilateral elements by joining the centroid of the surface with the mid points of the curves in the coordinate planes as shown in figure (2) [7, 8, 9].

Then each element is divided further into four elements by joining the centroid of that element with the mid–point of each side of the element. Thus one octant of the surface of the sphere is divided into 12 elements and the whole surface of the body is divided into 96 boundary elements. The above mentioned method is adopted in order to produce a uniform distribution of element over the surface of the body .

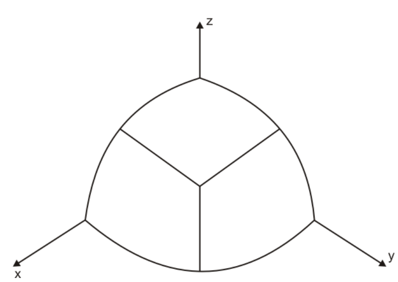

Figure (2)

Figure (3) shows the method for finding the coordinate (x_p, y_p, z_p) of any point P on the surface of the sphere.

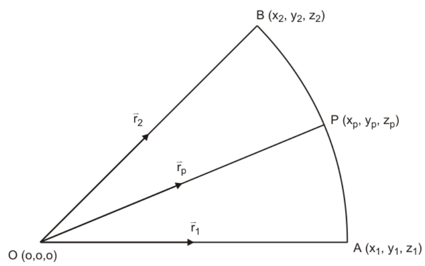

Figure (3)

From above figure, we have the following equation

$$|\vec{r}_p| = 1$$

$$\vec{r}_p \cdot \vec{r}_1 = \vec{r}_p \cdot \vec{r}_2$$

$$(\vec{r}_1 - \vec{r}_2) \cdot \vec{r}_p = 0$$

or in cartesian form

$$x_p^2 + y_p^2 + z_p^2 = 1$$

$$x_p (x_1 - x_2) + y_p (y_1 - y_2) + z_p (z_1 - z_2) = 0$$

$$x_p (y_1 z_2 - z_1 y_2) + y_p (x_2 z_1 - x_1 z_2) + z_p (x_1 y_2 - x_2 y_1) = 0$$

As the body possesses planes of symmetry, this fact may be used in the input to the program and only the non–redundant portion need be specified by input points. The other portions are automatically taken into account. The planes of symmetry are taken to be the coordinate planes of the reference coordinate system. The advantage of the use of symmetry is that it reduces the order of the resulting system of equations and consequently reduces the computing time in running a program. As a sphere is symmetric with respect to all three coordinate planes of the reference coordinate system, only one eighth of the body surface need be specified by the input points, while the other seven–eighth can be accounted for by symmetry.

The sphere is discretised into 96 and 384 boundary elements and the computed velocity distributions are compared with analytical solutions for the sphere using Fortran programming.

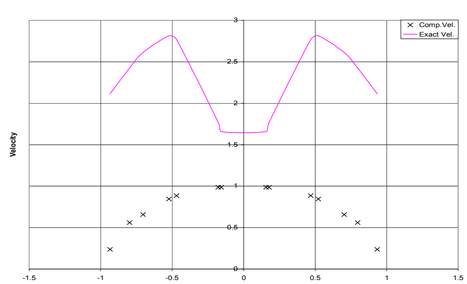

Figure (4): Comparison of computed and analytical velocity distributions over the surface of the sphere using 96 boundary elements.

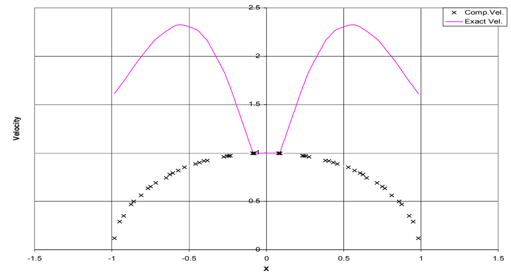

Figure (5): Comparison of computed and analytical velocity distributions over the surface of the sphere using 384 boundary elements.

Conclusion

Direct boundary element method has been applied to calculate the hyperbolic flow past a sphere. The improvement in results gained by taking 384 can be seen from figures (4) and (5) and such improvement increases with increase in number of boundary elements. Moreover, the computed results are in good agreement with exact results at the top of a body under consideration.

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