Direct Boundary Element Method for Calculation of Inviscid Compressible Flow past a Symmetric Aerofoil with Constant Element Approach

Muhammad Mushtaq[1], Nawazish Ali Shah[1], G. Muhammad[1], M.S. Khan[2] & F.H. Shah[3]

[1]Department of Mathematics, University of Engineering & Technology Lahore – 54890, Pakistan
[2]Department of Geological Engineering, University of Engineering & Technology Lahore – Pakistan
[3]Department of Mechanical Engineering, University of Engineering & Technology Lahore – Pakistan
mushtaqmalik2004@yahoo.co.uk

Abstract: In this paper, a direct boundary element method (DBEM) is applied to calculate the inviscid compressible flow past a symmetric aerofoil whereas in our previous papers, we applied indirect boundary element method (IBEM) for this purpose. The velocity distribution for the flow over the surface of the symmetric aerofoil has been calculated using direct constant boundary element approach. The accuracy of the computed results can be increased by increasing the number of boundary elements. The validity of this method is well checked by given tables and graphs.
[Muhammad Mushtaq, Nawazish Ali Shah, G. Muhammad, M.S. Khan and F.H. Shah. **Direct Boundary Element Method for Calculation of Inviscid Compressible Flow past a Symmetric Aerofoil with Constant Element Approach.** Life Sci J 2012;9(2):857-864]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 127

Keywords: Direct boundary element method, Inviscid Compressible flow, Velocity distribution, Symmetric aerofoil, Constant element.

1. Introduction

In the present period of science and technology, the popularity of boundary element methods rises for solving fluid flow problems and modeling physics in fluid. They provide the best base for the numerical methodology to solve the fluid flow problems. As well as providing the best solution of boundary integral equation based on a discretization process. The applications of boundary element methods rose on sound footing popular with the invention of electronic computer. The boundary element methods originated within the Department of Civil Engineering at Southampton University, U.K. These methods exist under different names such as panel methods, surface singularity method, boundary integral equation method, and boundary integral equation. First of all, finite difference method, finite element method, and finite volume method etc. were being used to solve numerically the problems in computational fluid dynamics. But later on, boundary element method has received much attention from the researchers due to its various advantages over the domain type methods. One of the advantages is that with boundary elements one has to discretize only the surface of the body, whereas with domain methods it is essential to discretize the entire region of the flow field. Moreover, this method is well–suited to problems with an infinite domain. The boundary element method can be classified into two categories i.e. direct and indirect. The direct method takes the form of a statement which provides the values of the unknown variables at any field point in terms of the complete set of all the boundary data. On the other hand, the indirect method utilizes a distribution of singularities over the boundary of the body and computes this distribution as the solution of integral equation. The equation of DBEM in the past can be formulated using either as an approach based on Green's theorem or a particular case of the weighted residual method in the past by many authors. (see Lamb, 1932; Ramsey, 1942, Milne-Thomson, 1968, Kellogge, 1929 and Brebbia and Walker, 1980). The direct and Indirect methods have been used in the past for flow field calculations around bodies (Morino 1975, Hess & Smith, 1967, Kohr, 2000, Luminita, 2008, Muhammad, 2009; Mushtaq, 2008, 2009, 2010, 2011& 2012). Most of the work on fluid flow calculations using boundary element methods has been done in the field of incompressible flow. Very few attempts have been made on flow field calculations using boundary element methods in the field of compressible flow. In this paper, the DBEM has been used for the solution of inviscid compressible flows around a symmetric aerofoil.

2. Mathematical Formulation

We know that equation of motion for two – dimensional, steady, irrotational, and isentropic flow (Mushtaq, 2010, 2011 & 2012) is

$$(1 - Ma^2)\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} = 0 \qquad (1)$$

where Ma is the Mach number and Φ is the total velocity potential of the flow. Here X and Y are the space coordinates.

Using the dimensionless variables, $x = X$, $y = \beta Y$, where $\beta = \sqrt{1 - Ma^2}$, equation (1) becomes

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2} = 0$$

or $\nabla^2 \Phi = 0$ (2)

which is Laplace's equation.

3. Symmetric Aerofoil

The Joukowski transformation

$$z = \zeta + \frac{a^2}{\zeta} \qquad (3)$$

transforms the circle shown in figure (1) in the ζ – plane on to symmetric aerofoil in the z-plane.

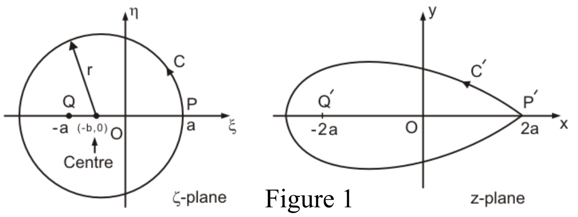


Figure 1

4. Flow Past a Symmetric Aerofoil

Consider the flow past a symmetrical aerofoil and let the onset flow be the uniform stream with velocity U in the positive direction of the x – axis as shown in figure (2) .

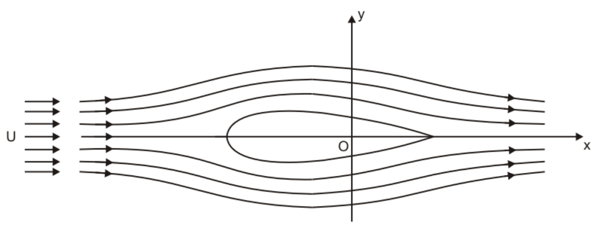


Figure 2: Flow past a symmetric aerofoil.

Exact Velocity

The magnitude of the exact velocity distribution over the boundary of a symmetric aerofoil is given by Chow [3] & Mushtaq [13, 17,18]

$$\text{as} \quad V = U \left| \frac{1-\left(\dfrac{r}{z-b}\right)^2}{1-\left(\dfrac{a}{z}\right)^2} \right|$$

where r = radius of the circular cylinder,
a = Joukowski transformation constant
and $b = a - r$ = x-coordinates of the centre of the circular cylinder

In Cartesian coordinates, we have

$V = U$

Error!

$$\times \frac{\sqrt{[(x^2+y^2)^2-a^2(x^2-y^2)]^2+4a^4x^2y^2}}{(x^2+y^2)^2-2a^2(x^2-y^2)+a^4}$$

Boundary Conditions

Now the condition to be satisfied on the boundary of a symmetric aerofoil is

$$\vec{V} \cdot \hat{n} = 0 \qquad (4)$$

where $\hat{n}$ is the unit normal vector to the boundary of the aerofoil .

Since the motion is irrotational

$$\vec{V} = -\nabla \Phi$$

where Φ is the total velocity potential . Thus equation (4) becomes

$$(-\nabla \Phi) \cdot \hat{n} = 0$$

$$\text{or} \quad \frac{\partial \Phi}{\partial n} = 0 \qquad (5)$$

Now the total velocity potential Φ is the sum of the perturbation velocity potential $\phi_{s.a}$ where the subscript s . a stands for symmetric aerofoil and the velocity potential of the uniform stream $\phi_{u.s}$.(Mushtaq, 2010 & 2011)

$$\text{i.e.} \quad \Phi = \phi_{u.s} + \phi_{s.a} \qquad (6)$$

$$\text{or} \quad \frac{\partial \Phi}{\partial n} = \frac{\partial \phi_{u.s}}{\partial n} + \frac{\partial \phi_{s.a}}{\partial n} \qquad (7)$$

From equations (5) and (7) , we get

$$\frac{\partial \phi_{s.a}}{\partial n} + \frac{\partial \phi_{u.s}}{\partial n} = 0$$

$$\text{or} \quad \frac{\partial \phi_{s.a}}{\partial n} = -\frac{\partial \phi_{u.s}}{\partial n} \qquad (8)$$

But the velocity potential of the uniform stream , given in Milne – Thomson [7], Shah [8], is

$$\phi_{u.s} = -U x \qquad (9)$$

then

$$\frac{\partial \phi_{u.s}}{\partial n} = -U \frac{\partial x}{\partial n}$$

$$= -U(\hat{n} \cdot \hat{i}) \qquad (10)$$

Thus from equations (8) and (10), we get

$$\frac{\partial u_{s.a}}{\partial n} = U(\hat{n} \cdot \hat{i}) \qquad (11)$$

Now from the figure (3)

$$\vec{A} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

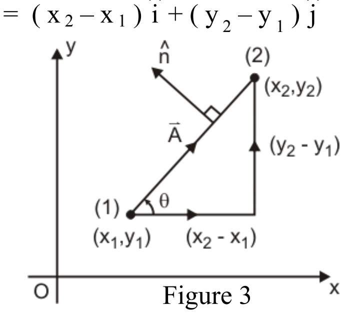


Figure 3

Therefore the unit vector in the direction of the vector $\vec{A}$ is given by

$$\hat{A} = \frac{(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

The outward unit normal vector $\hat{n}$ to the vector $\vec{A}$ is given by

$$\hat{n} = \frac{-(y_2 - y_1)\,\hat{n} + (x_2 - x_1)\,\hat{j}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Thus $$\hat{n}\cdot\hat{i} = \frac{(y_1 - y_2)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \qquad (12)$$

From equations (11) and (12) , we get

$$\frac{\partial \phi_{s.a}}{\partial n} = U\frac{(y_1 - y_2)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \qquad (13)$$

Equation (13) is the boundary condition which must be satisfied over the boundary of a symmetric aerofoil.

Equation of Direct Boundary Element Method

The equation of DBEM for two–dimensional flow [Mushtaq, 2008, 2009, 2010 & 2011] is :

$$-c_i\,\phi_i + \frac{1}{2\pi}\int_{\Gamma - i} \phi\frac{\partial}{\partial n}\left[\log\left(\frac{1}{r}\right)\right] d\Gamma + \phi_\infty$$

$$= \frac{1}{2\pi}\int_{\Gamma} \log\left(\frac{1}{r}\right)\frac{\partial \phi}{\partial n}\, d\Gamma \qquad (14)$$

where $c_i = 0$ when i is exterior to Γ

$= 1$ when i is interior to Γ

$= \frac{1}{2}$ when i lies on Γ and Γ is smooth.

Matrix Formulation

The equation (14) for the DBEM can be written in the discretized form as

$$-c_i\,\phi_i + \sum_{j=1}^{m}\left[\int_{\Gamma_j - i}\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log\frac{1}{r}\right)d\Gamma\right]\phi_j + \phi_\infty$$

$$= \sum_{j=1}^{m}\left[\int_{\Gamma_j}\left(\frac{1}{2\pi}\log\frac{1}{r}\right)d\Gamma\right]\frac{\partial \phi_j}{\partial n} \qquad (15)$$

The integrals in equation (15) on the elements can be calculated numerically except the element on which the fixed point 'i' is located. For this element the integrals are calculated analytically. Denoting the integrals on the L.H.S. of equation (15) by $\hat{H}_{ij}$ and that on the R.H.S. by G_{ij}, then

where $$\hat{H}_{ij} = \int_{\Gamma_j - i}\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log\frac{1}{r}\right)d\Gamma \qquad (16)$$

and $$G_{ij} = \int_{\Gamma_j}\frac{1}{2\pi}\log\left(\frac{1}{r}\right)d\Gamma \qquad (17)$$

For the case of that element on which the fixed point 'i' is lying, these integrals have been calculated.

Thus equation (15) can be written as

$$-c_i\,\phi_i + \sum_{j=1}^{m}\hat{H}_{ij}\,\phi_j + \phi_\infty = \sum_{j=1}^{m} G_{ij}\frac{\partial \phi_j}{\partial n} \qquad (18)$$

Defining

$$H_{ij} = \begin{cases} \hat{H}_{ij} & \text{when } i \neq j \\ \hat{H}_{ij} - c_i & \text{when } i = j \end{cases}$$

Equation (18) takes the form

$$\sum_{j=1}^{m} H_{ij}\,\phi_j + \phi_\infty = \sum_{j=1}^{m} G_{ij}\frac{\partial \phi_j}{\partial n}$$

which can be expressed in matrix form as

$$[H]\,\{\underline{U}\} = [G]\,\{\underline{Q}\} \qquad (19)$$

Since $\frac{\partial \phi}{\partial n}$ is specified at each node of the element, the values of the perturbation velocity potential ϕ are found at each node on the boundary via equation (19). The total potential Φ is then found from equation (6) which will then be used to calculate the velocity on the symmetric aerofoil.

The velocity midway between two nodes on the boundary can then be approximated by using the formula

$$\text{Velocity } \vec{V} = \frac{\Phi_{k+1} - \Phi_k}{\text{Length from node } k \text{ to } k+1} \qquad (20)$$

Process of Discretization

Now for the discretization of the boundary of the symmetric aerofoil, the coordinates of the extreme points of the boundary elements can be generated within computer programme using Fortran language as follows:

Divide the boundary of the circular cylinder into m elements in the clockwise direction by using the formula (Mushtaq 2009, 2010, 2011 & 2012).

$$\theta_k = [(m+3) - 2k]\frac{\pi}{m}, \quad k = 1, 2, \ldots\ldots, m \qquad (21)$$

Then the extreme points of these m elements of circular cylinder are found by

$$\xi_k = -b + r\cos\theta_k$$

$$\eta_k = r\sin\theta_k$$

Now by using Joukowski transformation in equation (3), the extreme points of the symmetric aerofoil are

$$x_k = \xi_k\left(1 + \frac{a^2}{\xi_k^2 + \eta_k^2}\right)$$

$$y_k = \eta_k\left(1 - \frac{a^2}{\xi_k^2 + \eta_k^2}\right)$$

where $k = 1, 2, \ldots\ldots, m$.

For constant boundary element approach there is only one node at the middle of the element and the potential ϕ and the potential derivative $\frac{\partial \phi}{\partial n}$ are constant over each element and equal to the value at the middle node of the element .

The coordinates of the middle node of each boundary element are given by

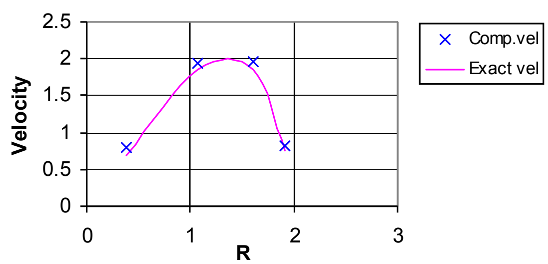
$$\left.\begin{aligned} x_m &= \frac{x_k + x_{k+1}}{2} \\ y_m &= \frac{y_k + y_{k+1}}{2} \end{aligned}\right\}$$

$$k, m = 1, 2, \ldots\ldots, n \qquad (22)$$

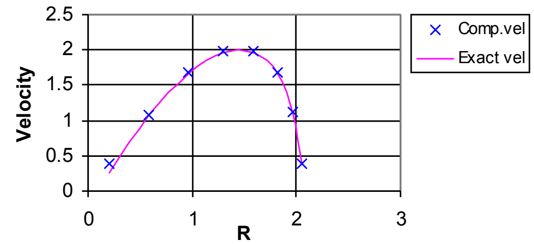
and therefore the boundary condition (13) in this case takes the form

$$\frac{\partial \phi_{s.a}}{\partial n} = U \frac{(y_1)_m - (y_2)_m}{\sqrt{[(x_2)_m - (x_1)_m]^2 + [(y_2)_m - (y_1)_m]^2}} \qquad (23)$$

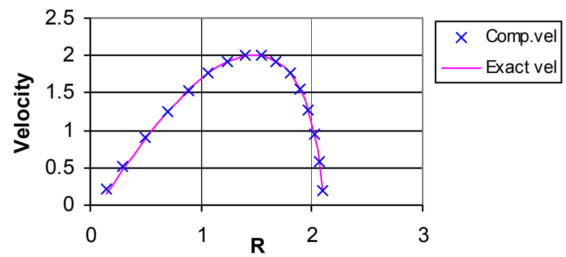
The following tables show the comparison of computed and analytical velocity distribution over the boundary of a symmetric aerofoil for 8, 16, 32, and 64 direct constant boundary elements.



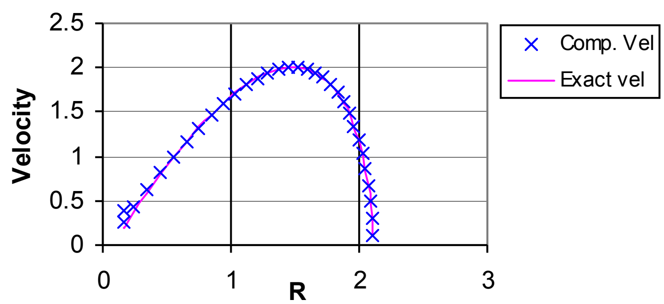
Graph 1: Comparison of computed and analytical velocity distributions over the boundary of a symmetric aerofoil using 8 boundary elements with direct constant element approach for r = 1.1, a = 0.1 and Ma = 0.7.



Graph 2: Comparison of computed and analytical velocity distributions over the boundary of a symmetric aerofoil using 16 boundary elements with direct constant element approach for r = 1.1, a = 0.1 and Ma = 0.7.



Graph 3: Comparison of computed and analytical velocity distributions over the boundary of a symmetric aerofoil using 32 boundary elements with direct constant element approach for r = 1.1, a = 0.1 and Ma = 0.7.



Graph 4: Comparison of computed and analytical velocity distributions over the boundary of a symmetric aerofoil using 64 boundary elements with direct constant element approach for r = 1.1, a = 0.1 and Ma = 0.7.

Table (1)

ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-1.87	.36	1.91	.80937E+00	.75969E+00
2	-1.36	.86	1.61	.19518E+01	.18480E+01
3	-.64	.86	1.07	.19386E+01	.18561E+01
4	-.13	.35	.38	.79640E+00	.68955E+00
5	-.13	-.35	.38	.79640E+00	.68955E+00
6	-.64	-.86	1.07	.19386E+01	.18561E+01
7	-1.36	-.86	1.61	.19518E+01	.18480E+01
8	-1.87	-.36	1.91	.80937E+00	.75969E+00

Table (2)

ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-2.04	.21	2.05	.39618E+00	.38702E+00
2	-1.88	.59	1.97	.11280E+01	.11044E+01
3	-1.59	.88	1.82	.16873E+01	.16594E+01
4	-1.21	1.03	1.59	.19881E+01	.19661E+01
5	-.80	1.03	1.30	.19835E+01	.19716E+01
6	-.42	.87	.96	.16713E+01	.16645E+01
7	-.12	.57	.58	.10856E+01	.10750E+01
8	.05	.19	.20	.38967E+00	.26843E+00
9	.05	-.19	.20	.38967E+00	.26843E+00
10	-.12	-.57	.58	.10856E+01	.10750E+01
11	-.42	-.87	.96	.16713E+01	.16645E+01
12	-.80	-1.03	1.30	.19835E+01	.19716E+01
13	-1.21	-1.03	1.59	.19881E+01	.19661E+01
14	-1.59	-.88	1.82	.16872E+01	.16594E+01
15	-1.88	-.59	1.97	.11280E+01	.11044E+01
16	-2.04	-.21	2.05	.39618E+00	.38702E+00

Table (3)

ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-2.09	.11	2.09	.19712E+00	.19455E+00
2	-2.05	.32	2.07	.58375E+00	.57600E+00
3	-1.97	.51	2.03	.94785E+00	.93637E+00
4	-1.85	.69	1.97	.12754E+01	.12622E+01
5	-1.70	.84	1.89	.15536E+01	.15410E+01
6	-1.52	.96	1.80	.17719E+01	.17620E+01
7	-1.32	1.04	1.68	.19216E+01	.19163E+01
8	-1.11	1.08	1.55	.19969E+01	.19969E+01
9	-.90	1.08	1.40	.19947E+01	.19999E+01
10	-.69	1.04	1.24	.19149E+01	.19236E+01
11	-.49	.95	1.07	.17602E+01	.17695E+01
12	-.31	.83	.89	.15357E+01	.15417E+01
13	-.16	.68	.70	.12489E+01	.12461E+01
14	-.04	.49	.49	.90793E+00	.88934E+00
15	.06	.28	.29	.52705E+00	.47740E+00
16	.12	.09	.15	.21584E+00	.15912E+00
17	.12	-.09	.15	.21584E+00	.15912E+00
18	.06	-.28	.29	.52705E+00	.47740E+00
19	-.04	-.49	.49	.90793E+00	.88934E+00
20	-.16	-.68	.70	.12489E+01	.12461E+01
21	-.31	-.83	.89	.15357E+01	.15417E+01

22	-.49	-.95	1.07	.17602E+01	.17695E+01
23	-.69	-1.04	1.24	.19149E+01	.19236E+01
24	-.90	-1.08	1.40	.19947E+01	.19999E+01
25	-1.11	-1.08	1.55	.19969E+01	.19969E+01
26	-1.32	-1.04	1.68	.19216E+01	.19163E+01
27	-1.52	-.96	1.80	.17719E+01	.17620E+01
28	-1.70	-.84	1.89	.15536E+01	.15410E+01
29	-1.85	-.69	1.97	.12754E+01	.12622E+01
30	-1.97	-.51	2.03	.94784E+00	.93637E+00
31	-2.05	-.32	2.07	.58375E+00	.57600E+00
32	-2.09	-.11	2.09	.19712E+00	.19455E+00

Table (4)

ELEMENT	X	Y	$R=\sqrt{X^2+Y^2}$	VELOCITY	EXACT VELOCITY
1	-2.10	.05	2.10	.98451E-01	.97672E-01
2	-2.09	.16	2.10	.29437E+00	.29110E+00
3	-2.07	.27	2.09	.48742E+00	.48207E+00
4	-2.04	.37	2.07	.67583E+00	.66862E+00
5	-2.00	.47	2.05	.85762E+00	.84901E+00
6	-1.95	.56	2.03	.10312E+01	.10216E+01
7	-1.89	.65	2.00	.11947E+01	.11847E+01
8	-1.82	.74	1.96	.13467E+01	.13367E+01
9	-1.74	.81	1.92	.14857E+01	.14763E+01
10	-1.66	.88	1.88	.16103E+01	.16021E+01
11	-1.57	.94	1.83	.17193E+01	.17127E+01
12	-1.47	.99	1.77	.18116E+01	.18072E+01
13	-1.37	1.03	1.72	.18863E+01	.18844E+01
14	-1.27	1.06	1.66	.19428E+01	.19435E+01
15	-1.17	1.08	1.59	.19804E+01	.19839E+01
16	-1.06	1.09	1.52	.19988E+01	.20051E+01
17	-.95	1.09	1.45	.19978E+01	.20065E+01
18	-.84	1.08	1.37	.19772E+01	.19882E+01
19	-.74	1.06	1.29	.19374E+01	.19501E+01
20	-.63	1.03	1.21	.18786E+01	.18922E+01
21	-.53	.98	1.12	.18014E+01	.18151E+01
22	-.44	.93	1.03	.17064E+01	.17193E+01
23	-.35	.87	.94	.15945E+01	.16052E+01
24	-.27	.80	.85	.14665E+01	.14739E+01
25	-.19	.72	.75	.13236E+01	.13260E+01
26	-.12	.64	.65	.11670E+01	.11624E+01
27	-.06	.55	.55	.99767E+00	.98398E+00
28	-.00	.45	.45	.81697E+00	.79117E+00
29	.04	.34	.34	.62619E+00	.58472E+00
30	.08	.23	.24	.42825E+00	.36984E+00
31	.12	.12	.17	.25123E+00	.18920E+00
32	.16	.03	.16	.38116E+00	.18610E+00
33	.16	-.03	.16	.38116E+00	.18610E+00
34	.12	-.12	.17	.25122E+00	.18920E+00
35	.08	-.23	.24	.42825E+00	.36984E+00
36	.04	-.34	.34	.62619E+00	.58472E+00
37	-.00	-.45	.45	.81697E+00	.79117E+00
38	-.06	-.55	.55	.99767E+00	.98398E+00
39	-.12	-.64	.65	.11670E+01	.11624E+01

40	-.19	-.72	.75	.13236E+01	.13260E+01
41	-.27	-.80	.85	.14665E+01	.14739E+01
42	-.35	-.87	.94	.15945E+01	.16052E+01
43	-.44	-.93	1.03	.17064E+01	.17193E+01
44	-.53	-.98	1.12	.18014E+01	.18151E+01
45	-.63	-1.03	1.21	.18786E+01	.18922E+01
46	-.74	-1.06	1.29	.19374E+01	.19501E+01
47	-.84	-1.08	1.37	.19772E+01	.19882E+01
48	-.95	-1.09	1.45	.19978E+01	.20065E+01
49	-1.06	-1.09	1.52	.19988E+01	.20051E+01
50	-1.17	-1.08	1.59	.19804E+01	.19839E+01
51	-1.27	-1.06	1.66	.19428E+01	.19435E+01
52	-1.37	-1.03	1.72	.18863E+01	.18844E+01
53	-1.47	-.99	1.77	.18116E+01	.18072E+01
54	-1.57	-.94	1.83	.17193E+01	.17127E+01
55	-1.66	-.88	1.88	.16103E+01	.16021E+01
56	-1.74	-.81	1.92	.14857E+01	.14763E+01
57	-1.82	-.74	1.96	.13467E+01	.13367E+01
58	-1.89	-.65	2.00	.11947E+01	.11847E+01
59	-1.95	-.56	2.03	.10312E+01	.10216E+01
60	-2.00	-.47	2.05	.85762E+00	.84901E+00
61	-2.04	-.37	2.07	.67583E+00	.66862E+00
62	-2.07	-.27	2.09	.48743E+00	.48207E+00
63	-2.09	-.16	2.10	.29438E+00	.29110E+00
64	-2.10	-.05	2.10	.98435E-01	.97670E-01

5. Conclusion

A direct boundary element method has been applied for the calculation of inviscid compressible flow past a symmetric aerofoil with constant element approach. The calculated flow velocities obtained using this method is compared with the analytical solutions for flow over the boundary of a symmetric aerofoil. It is found that from graphs 1 to 4, the computed results obtained by this method are good in agreement with the analytical ones for the body under consideration and the accuracy of the result increases due to increase of number of boundary elements.

6. Acknowledgement

We are thankful to the University of Engineering & Technology, Lahore – Pakistan for the financial support in my Ph. D research work. Also this research work is a part of my Ph.D thesis.

Corresponding author:
Dr. Muhammad Mushtaq,
Department of Mathematics,
University of Engineering & Technology,
Lahore – 54890, Pakistan.
mushtaqmalik2004@yahoo.co.uk

7. References

[1] Brebbia, C.A.: “The Boundary element Method for Engineers”, Pentech Press, 1978.
[2] Brebbia, C.A. and Walker, S.: “Boundary Element Techniques in Engineering” , Newness-Butterworths, 1980.
[3] C.Y. Chow, “An Introduction to Computational Fluid Mechanics”, John Wiiley & Sons, (1979).
[4] Kellogg, O.D.: “Foundations of Potential Theory”, Frederick Ungar Publishing Company (1929).
[5] Lamb, H.: “Hydrodynamics” 6th , Edition, Cambridge University Press,(1932).
[6] Ramsey, A.S.: “A Treatise on Hydrodynamics”, London G. Bell and Sons, Ltd. (1942).
[7] Milne-Thomson, L.M.: “Theoretical Hydrodynamics”, 5th Edition, London Macmillan & Co. Ltd., (1968).
[8] N.A. Shah, “Ideal Fluid Dynamics”, A–One Publishers, Lahore–Pakistan (2008).
[9] Morino, L., Chen, Lee-Tzong and Suciu,E.O.: “A steady and oscillatory subsonic and supersonic aerodynamics around complex configuration”, AIAA Journal, Vol. 13, No 3, pp 368-374 (1975).
[8] Hess, J.L. and Smith, A.M.O.: “Calculation of potential flow about arbitrary bodies”, Progress

in Aeronautical Sciences, Pergamon Press 1967, 8: 1-158, .

[10] Kohr ,M. : "A direct boundary integral method for a mobility problem", Georgian Mathematical Journal, Vol. 7, No. 1 (2000).

[11] Luminita G., Gabriela D., Mihai D., "Different Kinds of Boundary Elements for Solving the Problem of the Compressible Fluid Flow Around Bodies – a Comparison Study", Proceedings of the International Conference of Applied and Engineering Mathematics, 2008; 972–977.

[12] Luminita Grecu, "A Boundary Element Approach for the Compressible Flow Around Obstacles", Acta Universitatis Apulensis, Mathematics–Informatic No. 15/2008, 195–213.

[13] Mushtaq, M. Ph.D thesis "Boundary Element Methods for Compressible Fluid Flow Problems", University of Engineering & Technology, Lahore- Pakistan (2011).

[14] Muhammad, G., Shah, N.A., & Mushtaq, M.: "Indirect Boundary Element Methods for the Flow Past a Circular Cylinder with Constant Element Approach", Journal of Dohuk University, (Accepted for publication) (2009).

[15] Mushtaq, M., Shah, N.A., & Muhammad, G. "Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Circular Cylinder with Constant Element Approach", Journal of American Science, Vol. 5, No. 4, (2009) U.S.A.

[16] Mushtaq, M., Shah, N.A., & Muhammad, G. "Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Circular Cylinder with Linear Element Approach", Australian Journal of Basic and Applied Sciences, Vol. 2, No. 4, (2008) U.S.A.

[17] Mushtaq, M.,& Shah, N.A "Indirect Boundary Element Methods for the Calculation of Compressible Flow Past a Symmetric Aerofoil with Constant Element Approach", Journal of American Science, Vol. 6, No. 5, (2010) U.S.A.

[18] Mushtaq, M.,& Shah, N.A "Indirect Boundary Element Methods for the Calculation of Compressible Flow Past a Symmetric Aerofoil with Linear Element Approach Using Doublet Distribution Alone", Journal of American Science, Vol. 6, No. 11, (2010) U.S.A.

[19] Mushtaq, M.,& Shah, N.A "Indirect Boundary Element Methods for the Calculation of Compressible Flow Past a Joukowski Aerofoil with Constant Element Approach", Journal of American Science, Vol. 8, No. 2, (2012) U.S.A.

4/12/2012