Measures of Performance in the *M/M/1/N* Queue via the Methods of Order Statistics

Yousry H. Abdelkader[1] and A. I. Shawky[2]

[1]Math. Dept., Faculty of Science, Alexandria University, Egypt
[2]Dept. of Statistics, Faculty of Science, King Abdulaziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia
yabdelkader@yahoo.com

Abstract: This paper computes new measures of performance in Markovian queueing model with single-server and finite system capacity. The expected value and the variance of the minimum (maximum) number of customers in the system (queue) as well as the *r th* moments of the minimum (maximum) waiting time in the queue are derived. The computations of the proposed measures are depending on the methods of order statistics. The regular performance measures of the *M/M/1/N* model are considered as special cases of our results.

[Yousry H. Abdelkader and A. I. Shawky **Measures of Performance in the *M/M/1/N* Queue via the Methods of Order Statistics]** Life Science Journal. 2012; 9(1):945-953] (ISSN: 1097-8135). http://www.lifesciencesite.com. 138

Key Words: Queueing; Performance measures, Order statistics, Busy period, Waiting time, M/M/1/N queue.

1. Introduction

The study of queueing systems has often been concerned on the busy period and the waiting time, because they play a very significant role in the understanding of various queueing systems and their management. A busy period in a queuing system normally starts with the arrival of a customer who finds the system empty, and ends with the first time at which the system becomes empty again. Takagi and Tarabia (2009) provided an explicit probability density function of the length of a busy period starting with *i* customers for more general model *M/M/1/N*, where *N* is the capacity of the system, see also Tarabia (2001). Al Hanbali and Boxma (2010) studied the transient behaviour of a state dependent *M/M/1/N* queue during the busy period. Virtual waiting time at time *t* is defined to be the time that an imaginary customer would have to wait before service if they arrived in a queueing system at instant t. Gross and Harris (2003, Chapter 2) gave a derivation of the steady-state virtual waiting time distribution for an M/M/c model. Berger and Whitt (1995) provided various approximation and simulation techniques for several different queueing processes (waiting time, virtual waiting time, and the queue length). For more general queue models in waiting time, see Brandt and Brandt (2008). Limit theorems are proved by investigating the extreme values of the maximum queue length, the waiting time and virtual waiting time for different queue models. Serfozo (1987) discussed the asymptotic behavior of the maximum value of birth-death processes over large time intervals. Serfozo's results concerned on the transient and recurrent birth and death processes and related *M/M/c* queues. Asmussen (1998) introduced a survey of the present state of extreme value theory for queues and focused on the regenerative properties of queueing systems, which reduce the problem to study the tail of the maximum $\overline{X(\tau)}$ of the queueing process $\{X(t)\}$ during a regenerative cycle τ, where $\{X(t)\}$ is in discrete or continuous time. Artalejo *et al.* (2007) presented an efficient algorithm for computing the distribution for the maximum number of customers in orbit (and in the system) during a busy period for the *M/M/c* retrial queue. The main idea of their algorithm is to reduce the computation of the distribution of the maximum number of customers in orbit by computing certain absorption probabilities. Details about the extreme values in queues can be found in Park (1994) and Minkevičius (2009) and the references cited herein.

Our motivation is to obtain some complementary measures of performance of an *M/M/1/N* queue. The expected value and the variance of the minimum (maximum) number of customers in system (queue) as well as the *r th* moments of the minimum (maximum) waiting time will be discussed. This paper generalized the work of Abdelkader and Al-Wohaibi (2011). Their results can be obtained as special case of the results presented in this paper when $N \to \infty$.

Let us divide the number of arrival customers into *k* intervals and let X_j be the number of customers in each interval, the corresponding order statistics is defined by $X_{i:k}$. Three special cases are introduced: (a) *i*=*k*, defines the maximum number of customers presented in the system, (b) *i*=1, defines the minimum number of customers in the system, (c) *i*=*k*=1, defines the regular performance measures. So, our interest is to compute $\mu_{1:k} = E(X^{Min})$, $\sigma^2_{1:k} = Var(X^{Min})$, $\mu_{k:k} = E(X^{Max})$ and $\sigma^2_{k:k} = Var(X^{Max})$ where $X^{Min} = \underset{1\le j\le k}{Min}\{X_j\}$ and $X^{Max} = \underset{1\le j\le k}{Max}\{X_j\}$. Also, let T_i be the waiting time in the interval *i*. The expected value and the variance of the minimum (maximum) waiting times are computed respectively, as $\nu_{1:k} = E(T^{Min})$, $\alpha^2_{1:k} = Var(T^{Min})$, $\nu_{k:k} = E(T^{Max})$

and $\alpha_{k:k}^2 = Var(T^{Max})$, where $T^{Min} = \underset{1\le j\le k}{Min}\{T_j\}$ and $T^{Max} = \underset{1\le j\le k}{Max}\{T_j\}$, for $1 \le i \le k$.

2. Model and Description

Consider an M/M/1/N queue with arrival rate λ and service rate μ, where N denotes the capacity of he system with single server. Let $Q(t)$ be the number of customers in the system at time *t* .We define

$$p_n(t) = prob\{Q(t) = n|Q(0) = i\}.$$

Then the governing differential - difference equations for $p_N(t)$, the probability of having *n=N* customers in the system during *t*, for *h >0*, is given by

$$p_N(t+h) = p_N(t)(1-\mu h) + p_{N-1}(t)(\lambda h)(1-\mu h), n = N. \quad (1)$$

In steady-state equation for *M/M/1/N* queue representing this

$$-\rho\, p_0 + p_1 = 0, \qquad n = 0,$$
$$-(1+\rho)p_n + p_{n+1} + \rho p_{n-1} = 0,\ 0 < n < N,$$
$$-p_N + \rho\, p_{N-1} = 0, \qquad n = N.$$

The solution of the above equations is given by

$$p_n = \begin{cases} \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n, & \rho \ne 1 \\ \frac{1}{N+1}, & \rho = 1 \end{cases} \quad n = 0, 1, 2, \dots, N. \qquad (2)$$

Note that $\rho = \frac{\lambda}{\mu}$ need not be less than one because the number allowed in the system is controlled by the queue length (N-1), not by the rate of arrival and departure, λ and μ, respectively. It is easy to see that the effective rate $\lambda_{eff} = \lambda(1 - p_N)$.

Let Q be the number of customers in the system. Define the cumulative distribution function (cdf) of Q as:

$$F(x) = \Pr\{Q \le x\}$$
$$= \sum_{n=0}^{x} p_n,$$

Where x assumed to be integer.

For M/M/1/N queue, we have

$$F(x) = \begin{cases} \frac{1-\rho^{x+1}}{1-\rho^{N+1}}, & \rho \ne 1 \\ \frac{x+1}{N+1}, & \rho = 1. \end{cases} \qquad (3)$$

In the following, we state two of the needed theorems. These two theorems can be found in Arnold *et al.* (1992) and Barakat and Abdelkader (2004), respectively. the first theorem gives expressions for the first two moments of the *ith* order statistics, $X_{i:k}$, in a sample of size *k* in discrete case and the second theorem deals with the *rth* moments of $X_{i:k}$ in a continuous case.

Theorem 1. Let *S*, the support of the distribution, be a subset of non-negative integers. Then

$$E(X_{i:k}) = \mu_{i:k} = \sum_{x=0}^{\infty}[1 - F_{i:k}(x)]$$

and

$$E(X_{i:k}^2) = \mu_{i:k}^{(2)} = 2\sum_{x=0}^{\infty} x[1 - F_{i:k}(x)] + \mu_{i:k}.$$

whenever the moment on the left-hand side is assumed to exist.

Hence, the variance is given by

$$\sigma_{k:k}^2 = \mu_{k:k}^{(2)} - \mu_{k:k}^2.$$

In the case of independent identically distributed (*iid*) random variables, we have

$$\mu_{k:k} = \sum_{x=0}^{\infty}\left[1 - \left(F(x)\right)^k\right], \qquad (4)$$
$$\mu_{1:k} = \sum_{x=0}^{\infty}[1 - F(x)]^k, \qquad (5)$$
$$\mu_{k:k}^{(2)} = 2\sum_{x=0}^{\infty} x\left[1 - \left(F(x)\right)^k\right] + \mu_{k:k}, \qquad (6)$$
$$\mu_{1:k}^{(2)} = 2\sum_{x=0}^{\infty} x\,[1 - F(x)]^k + \mu_{1:k}, \qquad (7)$$
$$\sigma_{k:k}^2 = \mu_{k:k}^{(2)} - \mu_{k:k}^2, \qquad \sigma_{1:k}^2 = \mu_{1:k}^{(2)} - \mu_{1:k}^2, \qquad (8)$$

where $F_{k:k}(x) = [F(x)]^k$ and $F_{1:k}(x) = 1 - [1 - F(x)]^k$.

Theorem 2. *Let X_i, $1 \le i \le k$, be a non-negative r.v.'s with d.f.'s $F_i(x)$. Then, the rth moment of the ith order statistics in a sample of size k is given by*

$$\mu_{i:k}^{(r)} = r\int_0^{\infty} x^{r-1}\left(1 - F_{i:k}(x)\right)dx.$$

In case of *iid* random variables, when *i=k* and *i=1*, we get respectively

$$\mu_{k:k}^{(r)} = r\int_0^{\infty} x^{r-1}\left(1 - [F(x)]^k\right)dx, \qquad (9)$$
$$\mu_{1:k}^{(r)} = r\int_0^{\infty} x^{r-1}[1 - F(x)]^k dx. \qquad (10)$$

3. Performance measures

This section is devoted to introduce the proposed performance measures which are often useful for investigating the behavior of a queueing system. The mean and the variance of the minimum (maximum) number of customers in system as well as the mean and the variance of the minimum (maximum) waiting times are derived. The following lemma and consequence theorems give a procedure for the computations of these measures.

Lemma 1. Let X_i be *iid r.v.'s*, the *cdf* for the maximum, $F_{k:k}$, and the minimum, $F_{1:k}$, are given by

$$i)\quad F_{k:k}(x) = \frac{(1-\rho^{x+1})^k}{(1-\rho^{N+1})^k}, \qquad \rho \ne 1$$
$$= \frac{(x+1)^k}{(N+1)^k}, \qquad \rho = 1, \quad (11)$$

$$ii)\quad F_{1:k}(x)$$
$$= 1 - \frac{\rho^{(x+1)k}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\rho^{i(N-x)}, \quad \rho \ne 1$$
$$= 1 - \left(\frac{N-x}{N+1}\right)^k, \qquad \rho = 1. \quad (12)$$

Proof. From the definition of the *cdf* of $X_{k:k}$ and $X_{1:k}$, we have

$F_{k:k}(x) = [F(x)]^k,$

$F_{1:k}(x) = 1 - [1 - F(x)]^k.$

Plugging the value of $F(x)$ from (3) into the last two equations, we get (11) and (12). This completes the proof.■

3.1 The mean and the variance of the minimum and the maximum number of customers in the system

We are now ready to formulate our first result.

Theorem 3. The expected value and the variance of the maximum number of customers in the system are given by

$$\mu_{k:k} = N - \frac{1}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\rho^i(1-\rho^{iN})}{1-\rho^i},\ \rho \neq 1$$

$$= N - \frac{S(N,k)}{(N+1)^k}, \qquad \rho = 1, (13)$$

where ,

$$S(n,k) = \sum_{x=1}^{N} x^k, \qquad (14)$$

see, Calik *et al.* (2010) and Abramowitz and Stegun (1972).

The variance is then given by

$\sigma^2_{k:k} = \mu^{(2)}_{k:k} - \mu^2_{k:k}$,

where, at $\rho \neq 1$

$$\mu^{(2)}_{k:k} = N^2 - \frac{1}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\rho^i}{(1-\rho^i)^2}\{1 + \rho^i - (2N+1)\rho^{iN} + (2N-1)\rho^{i(N+1)}\},$$

and $\rho = 1$

$$\mu^{(2)}_{k:k} = N(N-1) - \frac{2[S(N,k+1) - S(N,k)]}{(N+1)^k} + N - \frac{S(N,k)}{(N+1)^k} = N^2 - \frac{2S(N,k+1) - S(N,k)}{(N+1)^k}.$$

Proof. Applying the definition

$$\mu_{k:k} = \sum_{x=0}^{N}(1 - F_{k:k}(x)) = \sum_{x=0}^{N}[1 - (F(x))^k]$$

$$= \sum_{x=0}^{N-1}[1 - \frac{(1-\rho^{x+1})^k}{(1-\rho^{N+1})^k}]$$

$$= N - \frac{1}{(1-\rho^{N+1})^k}\sum_{x=0}^{N-1}(1-\rho^{x+1})^k$$

$$= N - \frac{1}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\rho^i\sum_{x=0}^{N-1}(\rho^i)^x.$$

Expanding binomially and summing $\sum_{x=0}^{N-1}(\rho^i)^x = \frac{1-(\rho^i)^N}{1-\rho^i}$ we get the first part of (13).

For $\rho = 1$, using the definition of $\mu_{k:k}$ and the second part of equation (11), we get the second part of (13).

At $\rho \neq 1$:

$$\mu^{(2)}_{k:k} = 2\sum_{x=0}^{N} x(1 - F_{k:k}) + \mu_{k:k}$$

$$= 2\sum_{x=0}^{N} x\,[1 - (F(x))^k] + \mu_{k:k}$$

$$= 2\sum_{x=1}^{N-1} x\,[1 - (\frac{1-\rho^{x+1}}{1-\rho^{N+1}})^k] + \mu_{k:k}$$

$$= N(N-1) - \frac{2}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\sum_{x=1}^{N-1} x\,(\rho^{x+1})^i] + \mu_{k:k}$$

After some calculations we get

$$\mu^{(2)}_{k:k} = N^2 - \frac{1}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\rho^i}{(1-\rho^i)^2}.$$

$$\{1 + \rho^i - (2N+1)\rho^{iN} + (2N-1)\rho^{i(N+1)}\}$$

At $\rho = 1$:

$$\mu^{(2)}_{k:k} = 2\sum_{x=1}^{N-1} x\,[1 - \left(\frac{x+1}{N+1}\right)^k] + \mu_{k:k}$$

$$= N(N-1) - \frac{2}{(N+1)^k}\sum_{x=1}^{N-1} x(x+1)^k + \mu_{k:k}$$

$$= N(N-1) - \frac{2[S(N,k+1) - S(N,k)]}{(N+1)^k} + N - \frac{S(N,k)}{(N+1)^k} = N^2 - \frac{2S(N,k+1) - S(N,k)}{(N+1)^k}.■$$

Corollary 1. The expected value of the number of customers in the system is given by

$$L_s = \frac{\rho\{1-(N+1)\rho^N + N\rho^{N+1}\}}{(1-\rho)(1-\rho^{N+1})}, \qquad \rho \neq 1$$

$$= \frac{N}{2}, \qquad \rho = 1, \qquad (15)$$

which studied by Gross and Harris (2003) and Bhat (2008), and many text books in queueing theory.

Proof. Set k=1 in (13), we get

$$\mu_{1:1} = L_s = N - \frac{1}{1-\rho^{N+1}}\sum_{x=0}^{N-1} 1 - \rho^{x+1}$$

$$= N - \frac{1}{1-\rho^{N+1}}\left\{N - \frac{\rho(1-\rho^N)}{1-\rho}\right\}$$

After simple calculations we get the first part of equation (15). When $\rho = 1$, we have

$$\mu_{1:1} = L_s = N - \frac{S(N,1)}{N+1} = N - \frac{\sum_{x=1}^{N} x}{N+1} = \frac{N}{2}.■$$

Theorem 4. The expected value of the minimum number of customers in the system are given by

$$\mu_{1:k} = \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1-\rho^{iN}}{\rho^{iN}(1-\rho^i)}, \qquad \rho \neq 1,$$

$$\mu_{1:k} = \frac{S(N,k)}{(N+1)^k}, \qquad \rho = 1, \qquad (16)$$

where $k = 1,2,\ldots$ represent the busy period and $S(N,k)$ be defined in (14). To get the variance, it is enough to obtain the second moment and applying the definition in Theorem 1. Thus,

$$\mu_{1:k}^{(2)} = \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1+\rho^i-(2N+1)\rho^{iN}+(2N-1)\rho^{i(N+1)}}{\rho^{iN}(1-\rho^i)^2}, \quad \rho \neq 1$$

$$\mu_{1:k}^{(2)} = \frac{2[N\,S(N-1,k)-\,S(N-1,k+1)]+S(N,k)}{(N+1)^k}, \quad \rho = 1.$$

Proof. Applying the definition, at $\rho \neq 1$,

$$\mu_{1:k} = \sum_{x=0}^{N-1}\left(1-\frac{1-\rho^{x+1}}{1-\rho^{N+1}}\right)^k = \frac{1}{(1-\rho^{N+1})^k}\sum_{x=0}^{N-1}(\rho^{x+1}-\rho^{N+1})^k = \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1-\rho^{iN}}{\rho^{iN}(1-\rho^i)}. \quad \rho \neq 1$$

At $\rho = 1$, we have

$$\mu_{1:k} = \sum_{x=0}^{N-1}\left(1-\frac{x+1}{N+1}\right)^k = \frac{1}{(N+1)^k}\sum_{x=1}^{N}x^k = \frac{S(N,k)}{(N+1)^k}.$$

The second moment is given by, at $\rho \neq 1$,

$$\mu_{1:k}^{(2)} = 2\textstyle\sum_{x=0}^{N}x\left[1-F_{1:k(x)}\right] + \mu_{1:k}$$

$$= 2\textstyle\sum_{x=0}^{N}x[1-F(x)]^k + \mu_{1:k}$$

$$= 2\textstyle\sum_{x=1}^{N-1}x\left[1-\frac{1-\rho^{x+1}}{1-\rho^{N+1}}\right]^k + \mu_{1:k}$$

$$= \frac{2}{(1-\rho^{N+1})^k}\textstyle\sum_{x=1}^{N-1}x(\rho^{x+1}-\rho^{N+1})^k + \mu_{1:k}$$

$$= \frac{2\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{\rho^i\left[1-N\rho^{i(N-1)}+(N-1)\rho^{iN}\right]}{\rho^{iN}(1-\rho^i)^2} + \mu_{1:k}$$

$$= \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\textstyle\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1+\rho^i-(2N+1)\rho^{iN}+(2N-1)\rho^{i(N+1)}}{\rho^{iN}(1-\rho^i)^2}.$$

At $\rho = 1$, we have

$$\mu_{1:k}^{(2)} = 2\sum_{x=1}^{N-1}x\left(1-\frac{x+1}{N+1}\right)^k + \mu_{1:k} = \frac{2}{(N+1)^k}\sum_{x=1}^{N-1}x(N-x)^k + \mu_{1:k}$$

$$= \frac{2[N\,S(N-1,k)-\,S(N-1,k+1)]+S(N,k)}{(N+1)^k}.\blacksquare$$

3.2 The mean and the variance of the minimum and the maximum number of customers in the queue

Theorem 5. The expected value of the maximum number of customers in the queue are given by

$$\mu'_{k:k} = N-1-\frac{1}{(1-\rho^{N+1})^k}\textstyle\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\rho^{2i}(1-\rho^{i(N-1)})}{1-\rho^i}, \qquad \rho \neq 1,$$

$$\mu'_{k:k} = N-1-\frac{S(N,k)-1}{(N+1)^k}, \qquad \rho = 1, \tag{17}$$

where $k=1,2,\ldots$ represent the busy period and

$$S(N,k) = \textstyle\sum_{x=1}^{N}x^k.$$

To get the variance, it is enough to obtain the second moment and applying the definition in Theorem 1. Thus,

$$\mu'^{(2)}_{k:k} = N^2-1-\frac{1}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\rho^{2i}}{(1-\rho^i)^2}\{3-\rho^i+(2N+1)\rho^{i(N-1)}+(2N-1)\rho^{iN}\}, \quad \rho \neq 1$$

$$\mu'^{(2)}_{k:k} = N^2-1+\frac{1+S(N,k)-2S(N,k+1)}{(N+1)^k}, \quad \rho \neq 1.$$

Proof. Applying the definition

$$\mu'_{k:k} = \sum_{x=1}^{N}(1-F_{k:k}(x)) = \sum_{x=1}^{N}[1-(F(x))^k] = \sum_{x=1}^{N-1}[1-\frac{(1-\rho^{x+1})^k}{(1-\rho^{N+1})^k}]$$

$$= N-1-\frac{1}{(1-\rho^{N+1})^k}\textstyle\sum_{x=1}^{N-1}(1-\rho^{x+1})^k$$

$$= N-1-\frac{1}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\rho^i\sum_{x=1}^{N-1}(\rho^i)^x$$

Since $\sum_{x=1}^{N-1}(\rho^i)^x = \frac{\rho^i[1-(\rho^i)^{N-1}]}{1-\rho^i}$, we get the first part of (17).

For $\rho = 1$, using the definition of $\mu'_{k:k}$ and the second part of equation (11), we get the second part of (17). The second moment is given by

at $\rho \neq 1$,

$$\mu'^{(2)}_{k:k} = 2\sum_{x=1}^{N} x(1 - F_{k:k}) + \mu'_{k:k}$$

$$= 2\sum_{x=1}^{N} x\,[1 - (F(x))^k] + \mu'_{k:k}$$

$$= 2\sum_{x=1}^{N-1} x\,[1 - (\frac{1-\rho^{x+1}}{1-\rho^{N+1}})^k] + \mu'_{k:k}$$

$$= \mathrm{N(N-1)} - \frac{2}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\sum_{x=1}^{N-1} x\,(\rho^{x+1})^i] + \mu'_{k:k}$$

After some algebra, we obtain

$$\mu'^{(2)}_{k:k} = N(N-1) - \frac{2}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\rho^{2i}}{(1-\rho^i)^2}.$$

$$\{1 - N\rho^{i(N-1)} + (N-1)\rho^{iN}\} + \mu'_{k:k}$$

$$\mu'^{(2)}_{k:k} = N^2 - 1 - \frac{1}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{\rho^{2i}}{(1-\rho^i)^2}.$$

$$\{1 - \rho^i + (2N+1)\rho^{i(N-1)} + (2N-1)\rho^{iN}\}.$$

At $\rho = 1$:

$$\mu'^{(2)}_{k:k} = 2\sum_{x=1}^{N-1} x\left[1 - \frac{(x+1)^k}{(N+1)^k}\right] + \mu'_{k:k}$$

$$= N(N-1) - \frac{2}{(N+1)^k}\sum_{x=1}^{N-1} x(x+1)^k + \mu'_{k:k}$$

$$= N(N-1) - \frac{2[S(N,k+1) - S(N,k)]}{(N+1)^k} + \; N - 1 - \frac{S(N,k) - 1}{(N+1)^k}$$

$$= N^2 - 1 + \frac{1 + S(N,k) - 2S(N,k+1)}{(N+1)^k}.$$ ■

Corollary 2. The expected value of the number of customers in the queue is given by

$$L_q = \mu'_{1:1} = \frac{\rho^2[1 - N\rho^{N-1} + (N-1)\rho^N]}{(1-\rho)(1-\rho^{N+1})}, \quad \rho \neq 1$$

$$= \frac{N(N-1)}{2(N+1)}, \qquad \rho = 1. \qquad (18)$$

Proof. Set k=1 in (17), we get the relation (18).■

Theorem 6. The expected value of the minimum number of customers in the queue are given by

$$\mu'_{1:k} = \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1-\rho^{i(N-1)}}{\rho^{i(N-1)}(1-\rho^i)}, \quad \rho \neq 1,$$

$$\mu'_{1:k} = \frac{S(N-1,k)}{(N+1)^k}, \qquad \rho = 1, \qquad (19)$$

where $k=1,2,\ldots$ represent the busy period and $S(N,k)$ defined in (14).

The second moment is then given by

$$\mu'^{(2)}_{1:k} = \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{3 - (2N+1)\rho^{i(N-1)} + (2N-1)\rho^{iN}}{\rho^{i(N-1)}(1-\rho^i)^2}, \quad \rho \neq 1$$

$$\mu'^{(2)}_{1:k} = \frac{(2N+1)\,S(N-1,k) - 2\,S(N-1,k+1)}{(N+1)^k}, \quad \rho = 1$$

Proof. Applying the definition, at $\rho \neq 1$,

$$\mu'_{1:k} = \sum_{x=1}^{N}[1 - F(x)]^k = \sum_{x=0}^{N-1}\left(1 - \frac{1-\rho^{x+1}}{1-\rho^{N+1}}\right)^k$$

$$= \frac{1}{(1-\rho^{N+1})^k}\sum_{x=0}^{N-1}(\rho^{x+1} - \rho^{N+1})^k$$

$$\mu'_{1:k} = \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1-\rho^{i(N-1)}}{\rho^{i(N-1)}(1-\rho^i)}.$$

At $\rho = 1$, we have

$$\mu'_{1:k} = \sum_{x=1}^{N-1}\left(1 - \frac{x+1}{N+1}\right)^k$$

$$= \frac{1}{(N+1)^k}\sum_{x=1}^{N-1} x^k = \frac{S(N-1,k)}{(N+1)^k}.$$

The second moment is then given by, at $\rho \neq 1$,

$$\mu'^{(2)}_{1:k} = 2\sum_{x=0}^{N} x[1 - F_{1:k(x)}] + \mu'_{1:k}$$

$$= 2\sum_{x=0}^{N} x[1 - F(x)]^k + \mu'_{1:k}$$

$$= 2\sum_{x=1}^{N-1} x\left[1 - \frac{1-\rho^{x+1}}{1-\rho^{N+1}}\right]^k + \mu'_{1:k}$$

$$= \frac{2}{(1-\rho^{N+1})^k}\sum_{x=1}^{N-1} x(\rho^{x+1} - \rho^{N+1})^k + \mu'_{1:k}$$

$$= \frac{2\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{1 - N\rho^{i(N-1)} + (N-1)\rho^{iN}}{\rho^{i(N-1)}(1-\rho^i)^2} + \mu'_{1:k}$$

$$= \frac{\rho^{k(N+1)}}{(1-\rho^{N+1})^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\frac{3 - (2N+1)\rho^{i(N-1)} + (2N-1)\rho^{iN}}{\rho^{i(N-1)}(1-\rho^i)^2}.$$

At $\rho = 1$, we have

$$\mu_{1:k}^{'(2)} = 2\sum_{x=1}^{N-1} x\left(1-\frac{x+1}{N+1}\right)^k + \mu_{1:k}^{'}$$

$$= \frac{2}{(N+1)^k}\sum_{x=1}^{N-1} x(N-x)^k + \mu_{1:k}^{'}$$

$$= \frac{(2N+1)\,S(N-1,k) - 2\,S(N-1,k+1)}{(N+1)^k}, \quad \rho = 1$$

hence the proof. ■

4. Waiting time Distribution for M/M/1/N model

Other useful performance measures are the rth moments of the minimum and maximum waiting time in the queue. The cumulative probability distribution of the waiting time in the queue for the *M/M/1/N* queue is given by

$$W_q(t) = \begin{cases} 1-\frac{1-\rho}{1-\rho^N}\sum_{n=1}^{N-1}\rho^n\sum_{j=0}^{n-1}\frac{e^{-\mu t}(\mu t)^j}{j!}, & \rho \neq 1 \\ 1-\frac{1}{N}\sum_{n=1}^{N-1}\sum_{j=0}^{n-1}\frac{e^{-\mu t}(\mu t)^j}{j!}, & \rho = 1. \end{cases} \tag{20}$$

The expected waiting time is given by

$$W_q = \begin{cases} \dfrac{\rho[1-N\rho^{N-1}+(N-1)\rho^N]}{\mu(1-\rho)(1-\rho^N)}, & \rho \neq 1 \\ \dfrac{N-1}{2\mu}, & \rho = 1. \end{cases}$$

In the following two theorems are established. The first one deals with the rth moments of the minimum waiting time while the second deals with rth moments of the maximum waiting time.

Theorem 7. The rth moments of the minimum waiting time in the queue is given by

$$v_{1:k}^{(r)} = r\left(\frac{\rho}{1-\rho^N}\right)^k \sum_{i=0}^{k}(-1)^i\binom{k}{i}.\rho^{(N-1)i}\sum_{m=0}^{(k-i)(N-2)}\sum_{s=0}^{i(N-2)} a_{s,i}b_{m,k-i}\times\frac{\rho^m\Gamma(r+m+s)}{\mu^r k^{m+s+1}} \quad \rho \neq 1, \tag{21}$$

$$v_{1:k}^{(r)} = \frac{r}{N}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}.(N-1)^i\sum_{m=0}^{(k-i)(N-3)}\sum_{s=0}^{i(N-2)} d_{s,i}c_{m,k-i}\frac{\Gamma(r+m+s+k-i)}{\mu^r k^{r+m+s+k-i}}, \quad \rho = 1, \tag{22}$$

where $b_{m,k-i}$ is a coefficient of $(\lambda t)^m$ in the expansion $\left[\sum_{\ell=0}^{N-2}\frac{(\lambda t)^\ell}{\ell!}\right]^{k-i}$, i.e.,

$\left[\sum_{\ell=0}^{N-2}\frac{(\lambda t)^\ell}{\ell!}\right]^{k-i} = \sum_{m=0}^{(k-i)(N-2)} b_{m,k-i}\ (\lambda t)^m$,

$a_{s,i}$ is a coefficient of $(\mu t)^s$ in the expansion $\left[\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right]^i$, $c_{m,k-i}$ is a coefficient of $(\mu t)^m$ in the expansion $\left[\sum_{\ell=0}^{N-3}\frac{(\lambda t)^\ell}{\ell!}\right]^{k-i}$, and $d_{s,i}$ is a coefficient of $(\mu t)^s$ in the expansion $\left[\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right]^i$, see, Ahsanullah (1995) and Balakrishnan and Chan (1998).

Proof. Using equation (10) in Theorem 2 and the definition of $W_q(t)$ in equation (20) we have

$$v_{1:k}^{(r)} = r\int_0^\infty t^{r-1}\left[1-W_q(t)\right]^k dt$$

$$= r\int_0^\infty t^{r-1}\left(\frac{1-\rho}{1-\rho^N}\right)^k\left[\sum_{n=1}^{N-1}\rho^n\sum_{j=0}^{n-1}e^{-\mu t}\frac{(\mu t)^j}{j!}\right]^k dt$$

$$= r\left(\frac{1-\rho}{1-\rho^N}\right)^k\int_0^\infty t^{r-1}e^{-\mu kt}\left[\rho\sum_{n=1}^{N-1}\rho^{n-1}\sum_{j=0}^{n-1}\frac{(\mu t)^j}{j!}\right]^k dt$$

$$= r\left(\frac{\rho(1-\rho)}{1-\rho^N}\right)^k\int_0^\infty t^{r-1}e^{-\mu kt}\left[\sum_{j=0}^{N-2}\sum_{i=j}^{N-2}\frac{\rho^i(\mu t)^j}{j!}\right]^k dt$$

$$= r\left(\frac{\rho(1-\rho)}{1-\rho^N}\right)^k\int_0^\infty t^{r-1}e^{-\mu kt}\left[\sum_{j=0}^{N-2}\frac{\rho^j(\mu t)^j}{j!}.\frac{1-\rho^{N-1-j}}{1-\rho}\right]^k dt$$

$$= r\left(\frac{\rho}{1-\rho^N}\right)^k\int_0^\infty t^{r-1}e^{-\mu kt}\left[\sum_{j=0}^{N-2}\frac{(\lambda t)^j}{j!} - \rho^{N-1}\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right]^k dt$$

$$= r\left(\frac{\rho}{1-\rho^N}\right)^k\int_0^\infty t^{r-1}e^{-\mu kt}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left[\sum_{\ell=0}^{N-2}\frac{(\lambda t)^\ell}{\ell!}\right]^{k-i}.\rho^{(N-1)i}\left[\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right]^i dt$$

$$= r\left(\frac{\rho}{1-\rho^N}\right)^k\sum_{i=0}^{k}(-1)^i\binom{k}{i}.\rho^{(N-1)i}\sum_{m=0}^{(k-i)(N-2)}\sum_{s=0}^{i(N-2)} a_{s,i}b_{m,k-i}\,\lambda^m\mu^s\int_0^\infty t^{r+m+s-1}e^{-\mu kt}\,dt$$

$$v_{1:k}^{(r)} = r\left(\frac{\rho}{1-\rho^N}\right)^k\sum_{i=0}^{k}(-1)^i\binom{k}{i}.\rho^{(N-1)i}\sum_{m=0}^{(k-i)(N-2)}\sum_{s=0}^{i(N-2)} a_{s,i}b_{m,k-i}\frac{\rho^m\Gamma(r+m+s)}{\mu^r k^{m+s+1}}$$

$$= r\left(\frac{\rho}{1-\rho^N}\right)^k\sum_{i=0}^{k}(-1)^i\binom{k}{i}.\rho^{(N-1)i}\sum_{m=0}^{(k-i)(N-2)}\sum_{s=0}^{i(N-2)}\frac{a_{s,i}b_{m,k-i}\rho^m\Gamma(r+m+s)}{\mu^r k^{m+s+1}}, \quad \rho \neq 1.$$

At $\rho = 1$, from equation (10) and the second part of equation (20) we obtain

$$v_{1:k}^{(r)} = \frac{r}{N}\int_0^\infty t^{r-1}\left[\sum_{n=1}^{N-1}\sum_{j=0}^{n-1}e^{-\mu t}\frac{(\mu t)^j}{j!}\right]^k dt$$

$$= \frac{r}{N}\int_0^\infty t^{r-1}e^{-\mu kt}\left[\sum_{j=0}^{N-2}\sum_{i=j}^{N-2}\frac{(\mu t)^j}{j!}\right]^k dt$$

$$= \frac{r}{N}\int_0^\infty t^{r-1}e^{-\mu kt}\left[(N-1)\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!} - \mu t\sum_{\ell=0}^{N-3}\frac{(\mu t)^\ell}{\ell!}\right]^k dt$$

$$v_{1:k}^{(r)} = \frac{r}{N}\int_0^\infty t^{r-1}e^{-\mu kt}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}$$

$$\left[\mu t\ \sum_{\ell=0}^{N-3}\frac{(\mu t)^\ell}{\ell!}\ \right]^{k-i}(N-1)^i\left[\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\ \right]^i \mathrm{dt}$$

$$= \frac{r}{N}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}.(N-1)^i \times$$

$$\sum_{m=0}^{(k-i)(N-3)}\sum_{s=0}^{i(N-2)} d_{s,i}c_{m,k-i}\,\mu^{m+s+k-i}\times$$

$$\int_0^\infty t^{r+m+s+k-i-1}e^{-\mu kt}\,dt$$

$$v_{1:k}^{(r)} = \frac{r}{N}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}.(N-1)^i\sum_{m=0}^{(k-i)(N-3)}\sum_{s=0}^{i(N-2)} d_{s,i}c_{m,k-i}\,\mu^{m+s+k-i}\frac{\Gamma(r+m+s+k-i)}{\mu^r k^{r+m+s+k-i}}.$$

In particular, when r=k=1, from (21) we get

$$v_{1:1} = W_q = \frac{\rho}{1-\rho^N}\sum_{i=0}^{1}(-1)^i\binom{1}{i}.\rho^{(N-1)i}\times$$

$$\sum_{m=0}^{(1-i)(N-2)}\sum_{s=0}^{i(N-2)}\frac{a_{s,i}b_{m,1}-i\rho^m\Gamma(m+s+1)}{\mu\,k^{m+s+1}}$$

$$= \frac{\rho}{\mu(1-\rho^N)}\ [\sum_{m=0}^{N-2}a_{s,0}\,b_{m,}\rho^m\Gamma(m+1) -$$

$$\rho^{N-1}\sum_{s=0}^{N-2}a_{s,1}b_{0,0}\,\Gamma(s+1)],$$

$$b_{m,1} = \frac{1}{m!},\qquad a_{s,1} = \frac{1}{s!},\quad b_{0,0} = 1,\qquad a_{0,0} = 1$$

$$v_{1:1} = \frac{\rho}{\mu(1-\rho^N)}\ [\sum_{m=0}^{N-2}\rho^m - \rho^{N-1}\sum_{s=0}^{N-2}1]$$

$$= \frac{\rho}{\mu(1-\rho^N)}\left[\frac{1-\rho^{N-1}}{1-\rho} - (N-1)\rho^{N-1}\right]$$

$$= \frac{\rho}{\mu(1-\rho)(1-\rho^N)}\ [1 - N\rho^{N-1} + (N-1)\rho^N],\ \rho \neq 1.$$

Also, from (22), we obtain

$$v_{1:1} = W_q$$

$$= \frac{1}{N\mu}\left[-\sum_{m=0}^{N-3}d_{0,1}\,c_{m,1}\,\Gamma(m+1) + (N-1)\sum_{s=0}^{N-2}d_{s,1}c_{0,0}\,\Gamma(s+1)\right]$$

$$= \frac{N-1}{2\mu},\quad \rho = 1,$$

Such that $c_{m,1} = \frac{1}{m!}$, $d_{s,1} = \frac{1}{s!}$, $c_{0,0} = 1$, *and* $d_{0,1} = 1$.■

Theorem 8. The *r*th moments of the maximum waiting time in the queue is given by

$$v_{k:k}^{(r)} = r\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}.\frac{\rho^i}{(1-\rho^N)^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}.\rho^{(N-1)s}\times$$

$$\sum_{m=0}^{(i-s)(N-2)}\sum_{\eta=0}^{s(N-2)} a_{\eta,s}\,b_{m,i-s}\,\lambda^m\mu^\eta\frac{\Gamma(r+m+\eta)}{(\mu i)^{r+m+\eta}},$$

$$\rho \neq 1, \tag{23}$$

$$v_{k:k}^{(r)} = r\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}.\frac{1}{N^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}(N-1)^{i-s}\sum_{m=0}^{(i-s)(N-2)}\sum_{\eta=0}^{s(N-3)} g_{\eta,s}\,e_{m,i-s}\frac{\Gamma(r+m+\eta+s)}{\mu^r i^{r+m+\eta+s}},$$

$$\rho = 1, \tag{24}$$

where $b_{m,i-s}$ coefficient of $(\lambda t)^m$ in $\left(\sum_{\ell=0}^{N-2}\frac{(\lambda t)^\ell}{\ell!}\right)^{i-s}$, $a_{\eta,s}$ coefficient of $(\mu t)^\eta$ in $\left(\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right)^s$, $e_{m,i-s}$ coefficient of $(\mu t)^m$ in $\left(\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right)^{i-s}$ and $g_{\eta,s}$ coefficient of $(\mu t)^\eta$ in $\left(\sum_{\ell=0}^{N-3}\frac{(\lambda t)^\ell}{\ell!}\right)^s$.

Proof.

Using equation (9) in Theorem 2 and the definition of $W_q(t)$ in equation (20) we have

$$v_{k:k}^{(r)} = r\int_0^\infty t^{r-1}\left[1-(W_q(t))^k\right]dt$$

$$= r\int_0^\infty t^{r-1}\ \times$$

$$\left[1-\left(1-\frac{1}{1-P_N}\sum_{n=1}^{N-1}P_n\sum_{j=0}^{n-1}\frac{e^{-\mu t}\,(\mu t)^j}{j!}\right)^k\ \right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1-\left(1-\frac{1-\rho}{1-\rho^N}\sum_{n=1}^{N-1}\rho^n\sum_{j=0}^{n-1}e^{-\mu t}\frac{(\mu t)^j}{j!}\right)^k\right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1-\left(1-\frac{\rho(1-\rho)e^{-\mu t}}{1-\rho^N}\sum_{j=0}^{N-2}\sum_{i=j}^{N-2}\frac{\rho^i(\mu t)^j}{j!}\right)^k\right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1-\left(1-\frac{\rho(1-\rho)e^{-\mu t}}{1-\rho^N}\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}.\frac{\rho^j(1-\rho^{N-1-j})}{1-\rho}\right)^k\right]dt$$

$$v_{k:k}^{(r)} = r\int_0^\infty t^{r-1}\left[1-\left(1 - \frac{\rho e^{-\mu t}}{1-\rho^N}\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\cdot\rho^j(1-\rho^{N-1-j})\right)^k\right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1-\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left(\frac{\rho e^{-\mu t}}{1-\rho^N}\sum_{j=0}^{N-2}\frac{(\lambda t)^j}{j!}\cdot(1 - \rho^{N-1-j})\right)^i\right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1-\sum_{i=0}^{k}(-1)^i\binom{k}{i}\cdot\frac{\rho^i e^{-\mu i t}}{(1-\rho^N)^i}\left(\sum_{j=0}^{N-2}\frac{(\lambda t)^j}{j!}\cdot(1-\rho^{N-1-j})\right)^i\right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1-\sum_{i=0}^{k}(-1)^i\binom{k}{i}\cdot\frac{\rho^i e^{-\mu i t}}{(1-\rho^N)^i}\left(\sum_{\ell=0}^{N-2}\frac{(\lambda t)^\ell}{\ell!}-\rho^{N-1}\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right)^i\right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1 - \sum_{i=0}^{k}(-1)^i\binom{k}{i}\cdot\frac{\rho^i e^{-\mu i t}}{(1-\rho^N)^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}\cdot\left(\sum_{\ell=0}^{N-2}\frac{(\lambda t)^\ell}{\ell!}\right)^{i-s}\rho^{(N-1)s}\left(\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right)^s\right]dt$$

$$= r\int_0^\infty t^{r-1}[1-\sum_{i=0}^{k}(-1)^i\binom{k}{i}\cdot\frac{\rho^i e^{-\mu i t}}{(1-\rho^N)^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}\times \sum_{m=0}^{(i-s)(N-2)} b_{m,i-s}(\lambda t)^m\rho^{(N-1)s}\sum_{\eta=0}^{s(N-2)}a_{\eta,s}(\mu t)^\eta]\,dt,$$

where $b_{m,i-s}$ coefficient of $(\lambda t)^m$ in $\left(\sum_{\ell=0}^{N-2}\frac{(\lambda t)^\ell}{\ell!}\right)^{i-s}$ and $a_{\eta,s}$ coefficient of $(\mu t)^\eta$ in $\left(\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}\right)^s$. Thus

$$v_{k:k}^{(r)} = r\int_0^\infty t^{r-1}\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\cdot\frac{\rho^i e^{-\mu i t}}{(1-\rho^N)^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}\cdot\rho^{(N-1)s}\times \sum_{m=0}^{(i-s)(N-2)}\sum_{\eta=0}^{s(N-2)} a_{\eta,s}\,b_{m,i-s}\,\lambda^m\mu^\eta t^{m+\eta}\,dt$$

$$= r\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\cdot\frac{\rho^i}{(1-\rho^N)^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}\cdot\rho^{(N-1)s}\times \sum_{m=0}^{(i-s)(N-2)}\sum_{\eta=0}^{s(N-2)} a_{\eta,s}\,b_{m,i-s}\,\lambda^m\mu^\eta\frac{\Gamma(r+m+\eta)}{(\mu i)^{r+m+\eta}}, \rho\neq 1.$$

At $\rho = 1$, from equation (9) and the second equation of (20) we obtain

$$v_{1:k}^{(r)} = r\int_0^\infty t^{r-1}\left[1-\left(1-\frac{1}{N}\sum_{n=1}^{N-1}\sum_{j=0}^{n-1}e^{-\mu t}\frac{(\mu t)^j}{j!}\right)^k\right]dt$$

$$= r\int_0^\infty t^{r-1}\left[1-\left(1-\frac{e^{-\mu t}}{N}\sum_{j=0}^{N-2}\sum_{i=j}^{N-2}\frac{(\mu t)^j}{j!}\right)^k\right]dt$$

$$= r\int_0^\infty t^{r-1}\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\frac{e^{-i\mu t}}{N^i}\times \left((N-1)\sum_{j=0}^{N-2}\frac{(\mu t)^j}{j!}-\mu t\sum_{j=1}^{N-2}\frac{(\mu t)^{j-1}}{(j-1)!}\right)^i dt$$

$$= r\int_0^\infty t^{r-1}\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\frac{e^{-i\mu t}}{N^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}(\mu t)^s\times (N-1)^{i-s}\sum_{m=0}^{(i-s)(N-2)} e_{m,i-s}(\mu t)^m\sum_{\eta=0}^{s(N-3)} g_{\eta,s}(\mu t)^\eta\,dt$$

$$= r\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\cdot\frac{1}{N^i}\sum_{s=0}^{i}(-1)^s\binom{i}{s}\times (N-1)^{i-s}\sum_{m=0}^{(i-s)(N-2)}\sum_{\eta=0}^{s(N-3)} e_{m,i-s}\,g_{\eta,s}\frac{\Gamma(r+m+\eta+s)}{\mu^r i^{r+m+\eta+s}},\quad \rho = 1.$$

In particular, when r=k=1, from (23) we get

$$v_{1:1} = \frac{\rho}{\mu(1-\rho^N)}\left[\frac{1-\rho^{N-1}}{1-\rho}-(N-1)\rho^{N-1}\right] = \frac{\rho}{\mu(1-\rho)(1-\rho^N)}[1-N\rho^{N-1}+(N-1)\rho^N] = W_q,\quad \rho\neq 1.$$

Also, from (24), we obtain

$$v_{1:1} = \frac{1}{\mu N}\left\{(N-1)\sum_{m=0}^{N-2}e_{m,1}g_{0,0}\Gamma(m+1) - \sum_{\eta=0}^{N-3}e_{0,0}g_{\eta,0}\Gamma(\eta+2)\right\} = \frac{N-1}{2\mu} = W_q,\qquad \rho = 1,$$

such that $e_{0,0} = g_{0,0} = 1, e_{m,1} = \frac{1}{m!}$ and $g_{\eta,0} = \frac{1}{\eta!}$. ■

Since $\mu_{k:k}$ and $\mu'_{k:k}$ give the sum of the expected value of the maximum number of customers in the system and queue, we compute the expected value of the maximum number of customers in the system and queue during the busy period *k*, respectively, by

$$\Delta\mu_k = \mu_{k:k} - \mu_{k-1:k-1},$$
$$\Delta\mu'_k = \mu'_{k:k} - \mu'_{k-1:k-1},$$

with conventional $\mu_{0:k} = \mu'_{0:k} = 0$ and $\rho = \Delta\mu_k - \Delta\mu'_k$.

Similarly, the expected value of the maximum waiting time in the queue during the busy period *k* is computed by

$$\Delta v_k = v_{k:k} - v_{k-1:k-1}.$$

5. **Conclusion**

In this paper, we have considered the Markovian queueing model with a single-server and finite system capacity. The paper presents some complementary measures of performance which are depending on the methods of order statistics. The expected value and the variance of the minimum (maximum) number of customers in the system (queue) as well as the rth moments of the minimum (maximum) waiting time in the queue are derived. When $N \to \infty$ the results of Abdelkader and Al-Wohaibi (2011) can be obtained. Clearly, the expected value and the variance of the number of customers in the system (queue) as well as the rth moment waiting time can be obtained as special cases from our proposed measures when *k=1*. Although this work is currently restricted to the M/M/1/N model, it can be applied to other queueing models.

Corresponding author
Yousry H. Abdelkader
Math. Dept., Faculty of Science, Alexandria University, Egypt
yabdelkader@yahoo.com

References

1. Abdelkader, Y. H., Al-Wohaibi, M. (2011). Computing the performance measures in queueing models via the method of order statistics. Journal of Applied Mathematics, pp.1-12.
2. Abramowitz, M., Stegun, I. A. (1972). Handbook of Mathematical Functions. 2 nd. Ed., Dover Publications, New York.
3. Ahsanullah, M. (1995). Record Statistics. Nova Science Publishers, New York.
4. Al Hanbali, A. , Boxma, O. (2010). Busy period analysis of the state dependent *M/M/1/K* Queue. *Operations Research Letters*, 38(1), pp.:1-6.
5. Arnold, B.C., Balakrishnan, N., Nagaraja, H. N. (1992). *A First Course in Order Statistics.* John Wiley &Sons, New York.
6. Artalejo, J. R., Economou, A., Lopez-Herrero, M.J. (2007). Algorithmic analysis of the maximum queue length in a busy period for the *M/M/c* retrial queue. *INFORMS* Journal on Computing, 19(1), pp.: 121-126.
7. Asmussen, S.(1998). Extreme value theory for queues via cycle maxima. *Extremes*, 1(2), pp.: 137-168.
8. Balakrishnan, N., Chan, P. S. (1998). Log-gamma order statistics and linear estimation of parameters. In: Handbook of Statistics, Vol. 17, Elsevier Science, pp. 61-83.
9. Barakat, H. M., Abdelkader, Y. H. (2004). Computing the moments of order statistics from non-identical random variables. *Statistical Methods & Applications*, 13, pp.:15-26.
10. Berger, A. W. , Whitt, W. (1995). Maximum values in queueing processes. Probability in the Engineering and Informational Science, 9(3), pp.:375-409.
11. Bhat, U. N. (2008). An Introduction to Queueing Theory: Modeling and analysis in applications. Birkhauser, Boston.
12. Brandt, A., Brandt, M. (2008). Waiting time for M/M systems under state-dependent processor sharing, Queueing Systems 59(3-4) pp.:297-319.
13. Calik, S., Gungor, M., Colak, C. (2010). On the moments of order statistics from discrete distributions. *Pak. J. Statistics*, 26(2),pp.: 417-426.
14. Gross, D., Harris, C. (2003). *Fundamentals of Queuing theory*. 3rd Ed., Wiley Series in Probability and Statistics.
15. Minkevičius, S. (2009). On extreme value in open queueing networks. *Mathematical and Computer Modeling*, 50, pp.: 1058-1066.
16. Park,Y. S. (1994). Asymptotic distributions of maximum queue lengths for M/G/1 and GI/M/1 systems. *J. Korean Stat. Soc.*, 24, pp.: 19-29.
17. Serfozo, R. F. (1987). Extreme values of birth and death processes and queues. *Stochastic Processes and their Applications*, 27(C), pp: 291-306.
18. Takagi, H., Tarabia, A. (2009). Explicit probability density function for the length of a busy period in an M/M/1/K queue. *In: Advances in Queueing Theory and Network Applications*, Springer, New York, pp. 213-226 (Chapter 12).
19. Tarabia, A.M.K. (2001). A new formula for the transient behavior of a non-empty *M/M/1/*∞ queue. *Applied Mathematics and Computation*, 132, pp.1-10.