A Note on Characterization by Renewal Variable

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Abstract: The concept of renewal variable associated with a non-negative random variable X is used to identify the distribution of X as well as its failure rate. Some illustrated examples are given.

[Ali A. A- Rahman. **A Note on Characterization by Renewal Variable.** Life Science Journal 2011; 8(4):1118-1119]. (ISSN: 1097-8135). http://www.lifesciencesite.com. 137

Keywords: Characterization, Renewal process, Mean residual life, Failure rate, Exponential, Pareto and Pearsonian distributions.

Introduction

Let X be a non-negative random variable (often represents the life of a unit in a certain process)with finite mean μ, cdf $F_X(.)$ and survival function $\bar{F}_X(.)$. A new random variable Y with density function $f_Y(.)$ can be defined (see, e.g., **Cox** [1] as follows:

$$f_Y(x) = \frac{\bar{F}_X(x)}{\mu} \qquad (1.1)$$

The random variable Y has many applications in life length studies (see, e.g., **Scheaffer** [2]) as well as in renewal Process (**Zacks** [3]). **Gupta** [4] has shown that for large values of X, the random variable Y represents the life of the process when an operating component is replaced upon failure by another possessing the same life distribution. **Pakes and Khattree** [5] have demonstrated that Y is related to the length biased sampling.

Moreover, several authors have used Y to characterize some probability distributions. **Huang and Lin** [6] have characterized the exponential distribution using a relationship between the k^{th} moments of Y and X. **Gupta** [2] has given an explicit formula of the cdf of X in terms of the failure rate of Y.

The main objective of this note is to identify the distribution of X as well as its failure rate function in terms of the mean residual life and failure rate functions of Y.

1- The Main Result.

The following Theorem determines the survival function of X as well as its failure rate in terms of the mean residual life function of Y.

Theorem 2.1

Let X be a non-negative continuous random variable with finite mean μ, failure rate $r_X(.)$, density function $f_X(.)$ and survival function $\bar{F}_X(.)$. Denote by Y, its associated random variable defined by (1.1). Assume that $g_Y(.)$ is the mean residual life function of Y, then

$$\bar{F}_X(x) = \frac{c}{g_Y^2(x)}(1 + \dot{g}_Y(x))\exp\left(-\int \frac{dx}{g_Y(x)}\right), \qquad (2.1)$$

For some constant c to be determined from $\bar{F}_X(0) = 1$

$$r_X(x) = \frac{1}{g_Y(x)} + \frac{2\dot{g}_Y(x)}{g_Y(x)} - \frac{\ddot{g}_Y(x)}{1+\dot{g}_Y(x)} \qquad (2.2)$$

Proof. Using (1.1), the survival function $\bar{F}_Y(x)$, of Y will be:

$$\bar{F}_Y(x) = \mu^{-1}\int_x^\infty \bar{F}_X(z)\,dz \qquad (2.3)$$

By definition, the mean residual life function of Y (**Hall and Wellner** [7]) is given by:

$$g_Y(x) = \frac{\int_x^\infty \bar{F}_Y(z)dz}{\bar{F}_Y(x)} \qquad (2.4)$$

Using (2.3), we get:

$$g_Y(x) = \frac{\int_x^\infty \int_z^\infty \bar{F}_X(t)dt\,dz}{\int_x^\infty \bar{F}_X(t)dt}, \text{ i.e.,}$$

$$g_Y(x)\int_x^\infty \bar{F}_X(t)\,dt = \int_x^\infty \int_z^\infty \bar{F}_X(t)\,dt\,dz \qquad (2.5)$$

Differentiating (2.5) with respect to x, we get:

$$\frac{\bar{F}_X(x)}{\int_x^\infty \bar{F}_X(t)dt} = \frac{1+\dot{g}_Y(x)}{g_Y(x)} \qquad (2.6)$$

Integrating both sides of (2.6) with respect to x, we get:

$$\int_x^\infty \bar{F}_X(t)\,dt = \frac{c}{g_Y(x)}\exp\left(-\int \frac{dx}{g_Y(x)}\right) \qquad (2.7)$$

For some constant c to be determined using $\bar{F}_X(0) = 1$. Differentiating (2.7) with respect to x, we get:

$$\bar{F}_X(x) = \frac{c}{g_Y^2(x)}(1 + \dot{g}_Y(x))\exp\left(-\int \frac{dx}{g_Y(x)}\right) \qquad (2.8)$$

To prove the 2^{nd} result, take the logarithm of both sides of equation(2.8),we get:

$$\ln \bar{F}_X(x) = \ln c - 2\ln g_Y(x) + \ln(1 + \dot{g}_Y(x)) - \int \frac{dx}{g_Y(x)} \qquad (2.9)$$

Differentiating (2.9) with respect to x and recalling that

$r_X(x) = \frac{f_X(x)}{\bar{F}_X(x)}$

We get:

$r_X(x) = \frac{1}{g_Y(x)} + 2\,\frac{\dot{g}_Y(x)}{g_Y(x)} - \frac{\dot{g}_Y(x)}{1+\dot{g}_Y(x)}$

Our proof is complete.

Remark (2.1)

Denote by $r_Y(.)$ and $g_Y(.)$, the failure rate and mean residual life function of the renewal variable respectively and recalling that (**Ruiz and Navarro** [8]) :

$r_Y(x)\, g_Y(x) = 1 + \dot{g}_Y(x)$,

then the 2^{nd} part of Theorem (2.1) can be written as follows:

$r_X(x) = 2\, r_Y(x) - (g_Y(x))^{-1}\ (1 + \frac{\dot{g}_Y(x)}{r_Y(x)})$

Examples

(1) If $g_Y(x) = k$ = constant, then equation (2.1) gives:

$\bar{F}_X(x) = \exp(-\frac{x}{k})$, $x > 0$

Also equation (2.2) gives $r_X(x)$ =constant, which is the well known result of the exponential distribution.

(2) If $g_Y(x) = \frac{b+x}{\theta-2}$, $0 < a \le x$, $\theta > 2$

Then equations (2.1) and (2.2) give:

$F(x) = (\frac{a+b}{b+x})^{\theta}$, $0 < a \le x$ $r_X(x) = \frac{\theta}{b+x}$

Which are the well known results for the general Pareto distribution with parameters a, b, and θ.

(3) If $g_Y(x) = \frac{b-x}{\theta+1}$, $a < x < b$, $\theta > 0$.

Then equations (2.1) and (2.2) give:

$\bar{F}_X(x) =,\ (\frac{b-x}{b-a})^{\theta}$

$r_X(x) = \frac{\theta}{b-x}$

Which are the well known results for the 1^{st} type Pearsonian distribution with parameters a,b, and $\theta > 0$.

12/12/2012

Remarks

(1) Similar results for the uniform distribution with parameters a, b can be given. To see this, put θ = 1 in the last example.

(2) Similar results for the beta distribution with parameters 1,m can be obtained. To this end, put θ= m, b =1 and a= 0 in the last example.

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