Bayesian censored data viewpoint in Weibull distribution

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Abstract: The time of failure and average life of a component, measured from some specified time until it fails, is represented by a continuous random variable. Extensively in recent years, one distribution that has been used as a model to deal with such problems for product life is the Weibull distribution. The objective of this paper is to consider the estimation problem of the probability $P(Y < X)$ for Weibull distribution. Based on Classical Type I censored samples. The maximum-likelihood estimators (MLE) are obtained for stress–strength reliability. Bayes estimates under various loss functions are researched. Some computational results from intensive simulations are presented. In the end, in order to investigate the accuracy of estimations, an illustrative example is examined numerically by means of Monte Carlo simulation.

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1. Introduction

When modeling data under classical stress–strength analysis, the reliability (R) there is a system subject to a stress *Y* and strength *X*. Both *Y* and *X* are assumed random variables with known probability distributions. In this system, (R) represents the probability *(P)* that *Y* exceeds and *X*, i.e. $P(Y < X)$. This model has many applications in various areas. For example, if *Y* represents the maximum pressure caused by flooding and *X* represents the strength of the leg of a bridge on a stream, then *P* is the probability that the bridge will be solid. Another example, if *Y* and *X* represent the control and treatment groups respectively, then *P* measures the treatment effect. In this regard, the estimation of *P* will be important in making inferences.

Several authors have considered different choices for stress and strength distributions, including, [1]-[7].All of those choices were based on the complete sample study. The monograph by **Kotz *et al.*** [8] provided an excellent review of the development of this model. However, may not be unrealistic in experimenters often employ censoring schemes in life tests to shorten the test time or to reduce the test cost. However, censoring restricts the ability to exactly observe failure times. Two censored tests (type I and type II), are commonly used in industrial engineering applications. Type I censored test is conducted with a time censored scheme, in which the life test is terminated. The determination could be either occurs at time t_0 . Or certain number of failures before the time t_0 . Alternatively, the type II censored test is conducted with a failure censored scheme, in which the life test is terminated if the first *r* smallest failure lifetimes are collected, where the number of failure lifetimes is predetermined. Survived units at the termination time of test are considered as censoring**.** **Lawless** [9] provides detailed discussions for life tests with two censoring schemes, Based on censoring samples.

Although, extensive work has been done on the developments of the stress–strength models under

Complete samples, not much attention has been paid to when the data are censored (. Jiang and Wong [10] studied inference for stress-strength with truncated exponential distribution). Abd-Elfattah and Marwa ([11], [12]) studied inference for the stress- strength under exponentially and Weibull distributed type II censoring data. Recently, Statistical inference for the stress–strength parameter based on progressively type II exponentially censored data was discussed in Saracoglu, *et al*. [13]. While Lio and Tsai [14] studied $P(Y < X)$ under progressively first failure censored samples when X and Y have two parameters Burr type XII distributions.

In this paper, we consider the statistical inference of the reliability stress–strength parameter R = P(X < Y) when X and Y are independent Weibull random variables. Based on type-I censored Weibull distribution for both X and Y.

The comparison between Bayes estimate for R = P(Y < X) under LINEX and square error loss functions is the main target for this paper.

The rest of the paper organized as follows: In Section2, the Bayes estimate of R under mean square errors is derived. The Bayes estimate of R under LINEX loss function is provided in Section 3.The risk function is provided in Section 4. Simulation results and data analysis are presented in Section 5. A numerical example is given for illustration in Section 6. Some concluding remarks are given in Section 7.

2. Bayes Estimation Of θ_1 and θ_2 Under Type-I Censored Data

Recently, the Bayesian approach has received large attention for analyzing failure data or time-to-event data, and has been often proposed as a valid alternative to traditional statistical perspectives. The Bayesian approach to reliability analysis allows prior subjective knowledge on lifetime parameters and technical information on the failure mechanism, as well as experimental data, to be incorporated into the inferential procedure. Hence Bayesian methods usually require less sample data to achieve the same quality of inferences than methods based on sampling theory, which becomes extremely important in case of expensive testing procedures. In this section we are concerned on the estimation of the unknown parameters θ_1 and θ_2 of the Weibull distribution based on a type-I censored random sample of size n and m. Suppose that $X_1,....,X_n$ a random sample of n units is testes until the test is terminating at time Z_1. Times to failure for Q_1 observations are observed where Q_1 is random. Where these lifetimes observed only if $x_i \le Z_1$, $i = 1,..., n$. Therefore

$$\omega_i = \begin{cases} 1 & \text{if } x_i \le Z_1 \\ 0 & \text{if } x_i > Z_1 \end{cases} \qquad (1)$$

Where $Q_1 = \sum_{i=1}^{n} \omega_i$,in the case of $Y_1,...., Y_m$ the test will terminate at Z_2. Times to failure for Q_2 observations are observed where Q_2 is random. Where these lifetimes observed only when $x_j \le Z_2$, $j = 1,..., m$. Therefore

$$\varepsilon_j = \begin{cases} 1 & \text{if } y_j \le Z_2 \\ 0 & \text{if } y_j > Z_2 \end{cases} \qquad (2)$$

Where $Q_2 = \sum_{j=1}^{m} \varepsilon_j$,The likelihood function

$$L(x_1,....,x_n \mid \theta_1) = \frac{k^{Q_1}}{\theta_1^{Q_1}} \prod_{i=1}^{n} x_i^{(k-1)\omega_i} e^{\frac{-\sum_{i=1}^{n} x_i^k \omega_i}{\theta_1}} e^{\frac{-Z_1^k (n-Q_1)}{\theta_1}} \qquad (3)$$

and

$$L(y_1,..., y_m \mid \theta_2) = \frac{k^{Q_2}}{\theta_2^{Q_2}} \prod_{j=1}^{m} y_j^{(k-1)\varepsilon_j} e^{\frac{-\sum_{i=1}^{m} y_j^k \varepsilon_j}{\theta_2}} e^{\frac{-Z_2^k (m-Q_2)}{\theta_2}} \qquad (4)$$

Put $S_1 = \sum_{i=1}^{n} x_i^k \omega_i + Z_1^k (n - Q_1)$,

$S_2 = \sum_{j=1}^{m} y_j^k \varepsilon_j + Z_2^k (m - Q_2)$.

Because X_i, Y_j , $i = 1,...,n$ and $j = 1,...,m$ are independent and identically distributed Weibull random variables with Parameters θ_1 and θ_2 .
We will use linear transformation in some steps that to study the distributions of S_1 and S_2 , where X is Weibull random variable with parameter θ_1 , then

$$\psi = X^k \omega + Z_1^k (n - Q_1) \qquad (5)$$

equation (5) is distributed as exponential with parameter $\omega\theta_1$ to prove that, put $C = Z_1^k (n - Q_1)$ then

$$\psi = X^k \omega + C \qquad (6)$$

$$\text{and } \left(\frac{\psi - C}{\omega}\right)^{\frac{1}{k}} = X \qquad (7)$$

by using technique of linear transformation

$$\frac{dX}{d\psi} = \frac{1}{\omega_i k} \left(\frac{\psi - C}{\omega_i}\right)^{\frac{1}{k}-1} \qquad (8)$$

then

$$f(\psi) = \frac{1}{\theta_1 \omega_i} e^{-\frac{\psi - C}{\theta_1 \omega}} , \psi > C \qquad (9)$$

by substitute with the value of C in equation, then

$$f(\psi) = \frac{1}{\theta_1 \omega_i} e^{\frac{\psi - Z_1^k (n-Q_1)}{\theta_1 \omega_i}} , \psi > Z_1^k (n - Q_1) \qquad (10)$$

This has truncated exponential distribution with parameter $\omega\theta_1$.

So

$$\sum_{i=1}^{n} x_i^k \omega_i + Z_1^k (n - Q_1) \quad (11)$$

has gamma distribution with $(n+1, \omega_i \theta_1)$.

also, for the case Y ,

$$\sum_{j=1}^{m} y_j^k \varepsilon_j + Z_2^k (m - Q_2) \quad (12)$$

has gamma distribution with $(m+1, \varepsilon_j \theta_2)$.

Then, The distribution of S_1 and S_2 are:

$$g(S_1) = \frac{1}{\Gamma(n+1)(\omega_i \theta_1)^{n+1}} S_1^n e^{-\frac{S_1}{\omega_i \theta_1}}, S_1 > 0 \quad (13)$$

and

$$g(S_2) = \frac{1}{\Gamma(m+1)(\varepsilon_j \theta_2)^{m+1}} S_2^m e^{-\frac{S_2}{\varepsilon_j \theta_2}}, S_2 > 0 \quad (14)$$

Combining the prior distributions, with the likelihood functions using Bayes' theorem, the posterior density of θ_1 and θ_2 are as follows:

$$\pi_1(\theta_1 \mid x) = \frac{e^{-\frac{(a+S_1))}{\theta_1}} (a+S_1)^{(Q_1+c+1)}}{\theta_1^{Q_1+c} \Gamma(Q_1+c+1)} \quad (15)$$

$$\pi_2(\theta_2 \mid y) = \frac{e^{-\frac{(b+S_2)}{\theta_2}} (b+S_2)^{(Q_2+d+1)}}{\theta_2^{Q_2+d} \Gamma(Q_2+d+1)} \quad (16)$$

By using (15), (16) under squared error loss, the Bayes estimator of θ_1 and θ_2 are the posterior mean denoted by $\hat{\theta}_{MSE\,1}$ and $\hat{\theta}_{MSE\,2}$.

The posterior mean $E_{\pi_1}(\theta_1)$ and $E_{\pi_2}(\theta_2)$:

$$\hat{\theta}_{MSB\,1} = \frac{a + S_1}{Q_1 + c} \quad (17)$$

And

$$\hat{\beta}_{MSB\,2} = \frac{b + S_2}{Q_2 + d} \quad (18)$$

3. Bayes Estimator of θ_1 and θ_2 Based on LINEX Loss Function

The loss function plays a critical role in Bayesian perspective. Most authors use the simple quadratic (symmetric) loss function and obtain the posterior mean as the Bayesian estimate. However, in practice, the real loss function is often not symmetric. For example, Feynman [15] remarks that in the disaster of a space shuttle, the management may have overestimated the average life or reliability of solid fuel rocket booster. The consequences of overestimates, in loss of human life, are much more serious than the consequences of underestimates. In this case, an asymmetric loss function might be more appropriate. Varian [16] introduced LINEX (Linear-Exponential) loss function, which is the simple generalization of squared error (SE) loss function and cane be used in almost every situation (Zellner[18]).
The LINEX loss function is defined as follows:

$$L(\Lambda) = e^{u\Lambda} - u\Lambda - 1; u \neq 0 \, . \quad (19)$$

In this section, the Bayes estimators of θ_1 and θ_2 will be derived, using LINEX as follows: Since

$$f(x) = \frac{k}{\theta_1} x^{k-1} e^{-(x^k / \theta_1)}, \quad x > 0; k, \theta_1 > 0 \quad (20)$$

and

$$f(y) = \frac{k}{\theta_2} y^{k-1} e^{-(y^k / \theta_2)}, \quad y > 0; k, \theta_2 > 0 \quad (21)$$

are probability density function of X and Y.

Suppose $\Lambda_1 = \hat{\theta}_1 / \theta_1 - 1$, where $\hat{\theta}_1$ an estimate of θ_1 . Consider the following LINEX loss function.

$$L(\Lambda_1) = e^{u_1 \Lambda_1} - u_1 \Lambda_1 - 1; u_1 \neq 0 \quad (22)$$

The sign and magnitude of 'u_1' represent, respectively, the direction and degree of asymmetry. A positive value of 'u_1'is used when overestimation is more costly than underestimation; while a negative value of 'u_1' is used in the reverse situation for 'u_1' close to zero.

Also, the same definition for θ_2 where $\Lambda_2 = \hat{\theta}_2 / \theta_2 - 1$ and $\hat{\theta}_2$ is an estimate of θ_2. Again consider the following LINEX loss function.

$$L(\Lambda_2) = e^{u_2 \Lambda_2} - u_2 \Lambda_2 - 1; u_2 \neq 0 \quad (23)$$

The sign of u_2 treats like the sign of u_1.

Using the LINEX loss function (22), the posterior expectation of the loss function $L(\Lambda_1)$ with respect to $\pi_1(\theta_1 / x)$ for θ_1 is

$$E[L(\Lambda_1)] = \int_0^\infty (e^{u_1(\hat{\theta}_1/\theta_1 - 1)} - u_1(\frac{\hat{\theta}_1}{\theta_1} - 1) - 1)\pi_1(\theta_1 \mid x)d\theta_1$$

$$= e^{-u_1} E(e^{u_1(\hat{\theta}_1/\theta_1)}) - u_1 E(\frac{\hat{\theta}_1}{\theta_1}) - 1. \quad (24)$$

The value of $\hat{\theta}_1$ that minimizes the posterior expectation of the loss function $L(\Lambda_1)$ denoted by $\hat{\theta}_{LB1}$ is obtained by solving the following equation

$$\frac{\partial E[L(\Lambda_1)]}{\partial \hat{\theta}_1} = e^{-u_1} E(\frac{u_1}{\theta_1} e^{u_1(\hat{\theta}_1/\theta_1)}) - u_1 E(\frac{1}{\theta_1}) = 0 \quad (25)$$

That is, $\hat{\theta}_{LB1}$ is the solution of the equation:

$$E(\frac{1}{\theta_1} e^{u_1(\hat{\theta}_1/\theta_1)}) = e^{u_1} E(\frac{1}{\theta_1}) \quad (26)$$

Provided that all expectations exist and finite, we get the optimal estimate of $\hat{\theta}_1$ relative to $L(\Lambda_1)$

$$E(\frac{1}{\theta_1} e^{u_1(\frac{\hat{\theta}_{LB1}}{\theta_1})}) = \frac{(a+S_1)^{(Q_1+c+1)}}{\Gamma(Q_1+c+1)} \int_0^\infty \frac{e^{-\frac{(a+S_1)+u_1\hat{\theta}_1}{\theta_1}}}{\theta_1^{Q_1+c+1}} d\theta_{LB1} \quad (27)$$

$$= \frac{(a+S_1)^{(Q_1+c+1)}(Q_1+c+1)}{((a+S_1)+u_1\hat{\theta}_{LB1})^{Q_1+c+2}}$$

For the right hand side of equation (27)

$$e^{u_1} E(\frac{1}{\theta_1}) = e^{u_1} \int_0^\infty \frac{e^{-\frac{(a+S_1)}{\theta_1}} (a+S_1)^{(Q_1+c+1)}}{\theta_1^{Q_1+c+1}\Gamma(Q_1+c+1)} d\theta_1$$

$$= e^{u_1} \frac{(a+S_1)^{(Q_1+c+1)}(Q_1+c+1)}{(a+S_1)^{(Q_1+c+2)}} \quad (8)$$

$$= e^{u_1} \frac{(Q_1+c+1)}{(a+S_1)}$$

Now, from (18) and (19) in (28)

$$\frac{(a+S_1)^{(Q_1+c+1)}(Q_1+c+1)}{((a+S_1)+u_1\hat{\theta}_{LB1})^{Q_1+c+2}} = e^{u_1} \frac{(Q_1+c+1)}{(a+S_1)} \quad (29)$$

Then $\hat{\theta}_{LB1}$ can be expressed as follow

$$(a+S_1)^{Q_1+c+2} = e^{u_1}((a+S_1)+u_1\hat{\theta}_{LB1})^{Q_1+c+2} \quad (30)$$

then

$$\hat{\theta}_{LB1} = \frac{(a+S_1)}{u1}(1 - e^{-u_1/Q_1+c+2}). \quad (31)$$

Let the loss function of θ_2 is $L(\Lambda_2)$. The posterior expectation of the loss function $L(\Lambda_2)$ with respect to $\pi_2(\theta_2/y)$ of $\hat{\theta}_2$ is the value of $\hat{\theta}_2$ that minimizes the posterior expectation of the loss function $L(\Lambda_2)$ denoted by $\hat{\theta}_{LB2}$

$$\hat{\theta}_{LB2} = \frac{(b+S_2)}{u_2}(1 - e^{-u_2/Q_2+d+2}). \quad (32)$$

4. The Joint Risk Efficiency

The risk functions of estimators $\hat{\theta}_{LB1}$ and $\hat{\theta}_{LB2}$ which relative to $L(\Lambda_1)$ and $L(\Lambda_2)$ are of interest. These joint risk functions are denoted by $R_L(\hat{\theta}_{LB1}, \hat{\theta}_{LB2})$ and $R_L(\hat{\theta}_{MSB1}, \hat{\theta}_{MSB2})$, where subscript L denotes risk relative to $L(\Lambda_1)$ and $L(\Lambda_2)$. Let us study the joint risk efficiency of $\hat{\theta}_{LB1}$ and $\hat{\theta}_{LB2}$, we will find $R_L(\hat{\theta}_{LB1}, \hat{\theta}_{LB2})$ from equations (31)and (32) as the following:

$$R_L(\hat{\theta}_{LB1},\hat{\theta}_{LB2}) = [e^{-u_1}\int_0^\infty e^{u_1\hat{\theta}_{LB1}/\theta_1} g(S_1)dS_1 - \int_0^\infty u_1 \frac{\hat{\theta}_{LB1}}{\theta_1} g(S_1)dS_1 + \int_0^\infty u_1 g(S_1)dS_1 - \int_0^\infty u_1 g(S_1)dS_1]$$
$$[e^{-u_2}\int_0^\infty e^{u_2\hat{\theta}_{LB2}/\theta_2} g(S_2)dS_2 - \int_0^\infty u_2 \frac{\hat{\theta}_{LB2}}{\theta_2} g(S_2)dS_2 + \int_0^\infty u_2 g(S_2)dS_2 - \int_0^\infty u_2 g(S_2)dS_2]. \tag{33}$$

$$R_L(\hat{\theta}_{LB1},\hat{\theta}_{LB2}) = [\frac{e^{\frac{a(1-e^{\frac{-u_1}{Q_1+c+2}})}{\theta_1}-u_1}}{\Gamma(n+1)\theta_1^{n+1}\omega_i^{n+1}} \int_0^\infty e^{\frac{-S_1(1-(1-e^{\frac{-u_1}{Q_1+c+2}})\omega_i)}{\omega_i\theta_1}} S_1^n dS_1 - \frac{a(1-e^{\frac{-u_1}{Q_1+c+2}})}{\Gamma(n+1)\theta_1^{n+2}\omega_i^{n+1}} \int_0^\infty S_1^n e^{-\frac{S_1}{\omega_i\theta_1}} dS_1 - \frac{(1-e^{\frac{-u_1}{Q_1+c+2}})}{\Gamma(n+1)\theta_1^{n+2}\omega_i^{n+1}} \int_0^\infty S_1^{n+1} e^{-\frac{S_1}{\omega_i\theta_1}} dS_1 + u_1 - 1]$$
$$[\frac{e^{\frac{b(1-e^{\frac{-u_1}{Q_1+c+2}})}{\theta_2}-u_2}}{\Gamma(m+1)\theta_2^{m+1}\varepsilon_j^{m+1}} \int_0^\infty e^{\frac{-S_2(1-(1-e^{\frac{-u_2}{Q_2+d+2}})\omega_i)}{\varepsilon_j\theta_2}} S_2^m dS_2 - \frac{b(1-e^{\frac{-u_2}{Q_2+d+2}})}{\Gamma(m+1)\theta_2^{m+2}\varepsilon_j^{m+1}} \int_0^\infty S_2^m e^{-\frac{S_2}{\varepsilon_j\theta_2}} dS_2 - \frac{(1-e^{\frac{-u_2}{Q_2+d+2}})}{\Gamma(m+1)\theta_2^{m+1}\varepsilon_j^{m+1}} \int_0^\infty S_2^{m+1} e^{-\frac{S_2}{\varepsilon_j\theta_2}} dS_2 + u_2 - 1]. \tag{34}$$

Then

$$R_L(\hat{\theta}_{LB1},\hat{\theta}_{LB2}) = [\frac{e^{\frac{a(1-e^{\frac{-u_1}{Q_1+c+2}})}{\theta_1}-u_1}}{(1+(1-e^{\frac{-u_1}{Q_1+c+2}})\omega_i)^{n+1}} - (1-e^{\frac{-u_1}{Q_1+c+2}})(a\theta_1^{-1}+(n+1))+u_1-1]$$
$$[\frac{e^{\frac{b(1-e^{\frac{-u_2}{Q_2+d+2}})}{\theta_2}-u_2}}{(1+(1-e^{\frac{-u_2}{Q_2+d+2}})\varepsilon_j)^{m+1}} - (1-e^{\frac{-u_2}{Q_2+d+2}})(b\theta_2^{-1}+(m+1))+u_2-1]. \tag{35}$$

In the same manner, we get

$$R_L(\hat{\theta}_{MSB1},\hat{\theta}_{MSB2}) = E_{XY}(L(\Lambda_1),L(\Lambda_2))$$
$$= \int_0^\infty\int_0^\infty L(\Lambda_1)L(\Lambda_2)g(S_1)g(S_2)dS_1dS_2$$
$$= \int_0^\infty [e^{u_1(\hat{\theta}_{MSB1}/\theta_1-1)} - u_1(\frac{\hat{\theta}_{MSB1}}{\theta_1}-1)-1]g(S_1)dS_1$$
$$\int_0^\infty [e^{u_2(\hat{\theta}_{MSB2}/\theta_2-1)} - u_2(\frac{\hat{\theta}_{MSB2}}{\theta_2}-1)-1]g(S_2)dS_2. \tag{36}$$

Then

$$R_L(\hat{\theta}_{MSB1},\hat{\theta}_{MSB2}) = [e^{-u_1}\int_0^\infty e^{u_1(\hat{\theta}_{MSB1}/\theta_1)} g(S_1)dS_1 - u_1\int_0^\infty \frac{\hat{\theta}_{MSB1}}{\theta_1} g(S_1)dS_1 + u_1 - 1]$$
$$[e^{-u_2}\int_0^\infty e^{u_2(\hat{\theta}_{MSB2}/\theta_2)} g(S_2)dS_2 - u_2\int_0^\infty \frac{\hat{\theta}_{MSB2}}{\theta_2} g(S_2)dS_2 + u_2 - 1]. \tag{37}$$

by using the equations (13),(14),(17) and (18) in(37)

$$R_L(\hat{\theta}_{MSB1},\hat{\theta}_{MSB2})=\Bigg[\frac{e^{u_1\left(\frac{a}{\theta_1(Q_1+c)}-1\right)}}{\Gamma(n+1)\omega_i^{n+1}\theta_1^{n+1}}\int_0^\infty S_1^n e^{S_1\left(\frac{u_1\omega_i-(Q_1+c)}{\theta_1(Q_1+c)\omega_i}\right)}dS_1-\frac{u_1 a}{\Gamma(n+1)(Q_1+c)\theta_1^{n+2}\omega_i^{n+1}}$$

$$\int_0^\infty S_1^n e^{-\frac{S_1}{\omega_i\theta_1}}dS_1-\frac{u_1}{\Gamma(n+1)(Q_1+c)\theta_1^{n+2}\omega_i^{n+1}}$$

$$\int_0^\infty S_1^{n+1} e^{-\frac{S_1}{\omega_i\theta_1}}dS_1+u_1-1\Bigg]\Bigg[\frac{e^{u_2\left(\frac{b}{\theta_2(Q_2+d)}-1\right)}}{\Gamma(m+1)\varepsilon_j^{m+1}\theta_2^{m+1}}$$

$$\int_0^\infty S_2^m e^{S_2\left(\frac{u_2\varepsilon_j-(Q_2+d)}{\theta_2(Q_2+d)\varepsilon_j}\right)}dS_2-\frac{u_2 b}{\Gamma(m+1)(Q_2+d)\theta_2^{m+2}\varepsilon_j^{m+1}}$$

$$\int_0^\infty S_2^m e^{-\frac{S_2}{\varepsilon_j\theta_2}}dS_2-\frac{u_2}{\Gamma(m+1)(Q_2+d)\theta_2^{m+2}\varepsilon_j^{m+1}} \qquad (38)$$

$$\int_0^\infty S_2^{m+1} e^{-\frac{S_2}{\varepsilon_j\theta_2}}dS_2+u_2-1\Bigg].$$

Then

$$R_L(\hat{\theta}_{MSB1},\hat{\theta}_{MSB2})=\Bigg[\frac{(Q_1+c)^{n+1}e^{u_1(a/\theta_1(Q_1+c)-1)}}{((Q_1+c)-u_1\omega_i)^{n+1}}$$

$$-\frac{u_1}{(Q_1+c)}(a\theta_1^{-1}+(n+1))+u_1-1\Bigg]\Bigg[\frac{(Q_2+d)^{m+1}\varepsilon_j e^{u_2(b/\theta_2(Q_2+d)-1)}}{((Q_2+d)-u_2\varepsilon_j)^{m+1}} \qquad (39)$$

$$-\frac{u_2}{(Q_2+d)}(b\theta_2^{-1}+(m+1))+u_2-1\Bigg].$$

The risk efficiency of $R_L(\hat{\theta}_{LB1},\hat{\theta}_{LB2})$ with respect to $R_L(\hat{\theta}_{MSB1},\hat{\theta}_{MSB2})$ under LINEX Loss $L(\Lambda_1)$ and $L(\Lambda_2)$ may be defined as follows:

$$RE3_L(\hat{\theta}_{LB1},\hat{\theta}_{LB2},\hat{\theta}_{MSB1},\hat{\theta}_{MSB2})=\frac{R_L(\hat{\theta}_{MSB1},\hat{\theta}_{MSB2})}{R_L(\hat{\theta}_{LB1},\hat{\theta}_{LB2})}. \qquad (40)$$

4. Simulation Study

We needed to check whether an estimator is inadmissible under some loss function or not. If so, the estimator would not be used for the losses specified by that loss function. For this purpose the risks of the estimators and risk efficiency have been computed $RE3$. So we will obtain the MLE for the unknown parameters of the Weibull distribution then use them to obtain Bayes estimators under LINEX loss function and Bayes estimator under square errors .The following steps will be considered to obtain the estimators:

Step (1): Generate random samples $X_1,\dots,X_n$ from Weibull distribution with sample sizes 5, 10, 15 and 20, the shape parameters θ_1 , for each value of the sample size of n we will generate 1000 random samples from Weibull distribution in the case of time=30.

Step (2): Similarly, we can obtain the generate samples for $y_1,\dots,y_m$ from Weibull distribution with parameters θ_2 , for each value of the sample size of m we will generate 1000 random samples from Weibull distribution.

Step (3): Using the Equation (35) to find the $R_L(\hat{\theta}_{LB1},\hat{\theta}_{LB2})$ and use Equation (39) to find $R_L(\hat{\theta}_{MSB1},\hat{\theta}_{MSB2})$ and Finally, Supply the values of risk function $RE3_L$ from equation (40).

Step (4): we take the average of there 1000 values then calculate risk function $RE3_L$.

Step(5): repeat steps (1-4)in the case of time=50.

Table (1)_ $u_1 = 0.8, u_2 = 0.8, \theta_1 = 2, \theta_2 = 2$ *and* $R = 0.4$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	1.784	2.975	1.438	2.399	2.05E-03	8.38E-04	0.409
(10,10)	2.498	2.93	2.153	2.526	8.03E-04	3.98E-04	0.496
(15,15)	2.651	2.767	2.367	2.471	4.12E-04	2.32E-04	0.562
(20,20)	2.587	3.433	2.36	3.132	2.472E-04	1.51E-04	0.613
(5,4)	2.079	2.88	1.676	2.56	2.32E-03	9.34E-04	0.404
(10,8)	2.095	2.426	1.805	2.547	9.552E-04	4.62E-04	0.484
(10,9)	2.16	2.397	1.862	2.34	8.72E-04	4.27E-04	0.49
(20,18)	2.134	2.766	1.947	2.504	2.73E-04	1.64E-04	0.603
(20,19)	2.1	3.092	1.99	2.811	2.591E-04	1.5E-04	0.608

Table (2) $u_1 = 0.8, u_2 = 0.8, \theta_1 = 1, \theta_2 = 2$ *and* $R = 0.333$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	1.136	1.597	0.916	1.45	5.29E-04	5.40E-04	1.017
(10,10)	1.33	2.371	1.146	2.044	3.37E-04	3.41E-04	1.011
(15,15)	1.283	1.089	1.145	1.698	2.067E-04	2.076E-04	1.004
(20,20)	1.12	2.32	0.998	2.119	1.4E-04	1.4E-04	1.001
(5,4)	0.688	1.566	0.555	1.78	7.058E-04	5.905E-04	0.978
(10,8)	1.314	1.864	1.132	2.042	3.945E-04	3.831E-04	0.971
(10,9)	1.22	1.805	1.052	1.541	3.637E-04	3.606E-04	0.992
(20,18)	1.1	2.115	1.21	1.915	1.522E-04	1.499E-04	0.985
(20,19)	1.325	1.95	1.013	2.285	1.453E-04	1.443E-04	0.993

Table (3) $u_1 = 1, u_2 = 1, \theta_1 = 1, \theta_2 = 2$ *and* $R = 0.333$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	1.273	2.47	1.167	1.975	1.54E-03	1.60E-03	1.045
(10,10)	1.254	2.192	1.037	1.813	1.13E-03	2.00E-03	1.764
(15,15)	1.626	2.247	1.417	1.958	5.35E-04	9.58E-04	1.51
(20,20)	1.3	2.233	1.18	2.005	4.05E-04	5.60E-04	1.381
(5,4)	0.982	2.077	0.785	2.328	1.811E-03	3.397E-03	1.005
(10,8)	1.474	2.104	1.136	2.001	9.589E-04	9.527E-04	0.993
(10,9)	1.609	1.836	1.379	1.557	8.842E-04	8.974E-04	1.015
(20,18)	1.325	2.363	1.244	2.131	3.7E-04	3.705E-04	1.001
(20,19)	1.415	2.571	1.324	2.328	3.5E-04	3.567E-04	1.01

Table (4) $u_1 = -1, u_2 = -1, \theta_1 = 1, \theta_2 = 2$ *and* $R = 0.333$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	1.156	1.764	1.004	1.533	1.55E-03	1.38E-03	0.893
(10,10)	1.356	1.957	1.232	1.779	8.51E-04	7.63E-04	0.896
(15,15)	1.256	2.212	1.02	2.057	5.23E-04	4.75E-04	0.909
(20,20)	1.114	2.6	1.12	2.453	3.50E-04	3.22E-04	0.919
(5,4)	1.343	1.652	1.167	1.417	1.744E-03	1.507E-03	0.819
(10,8)	1.12	2.254	1.245	2.02	1.002E-03	8.691E-04	0.867
(10,9)	1.364	2.304	1.232	2.094	9.205E-04	8.126E-04	0.883
(20,18)	1.17	2.554	1.017	2.603	3.845E-04	3.489E-04	0.908
(20,19)	1.253	2.664	1.24	2.507	3.667E-04	3.350E-04	0.914

Table (5) $u_1 = 0.8, u_2 = 0.8, \theta_1 = 2, \theta_2 = 3$ *and* $R = 0.4$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	1.502	2.49	1.835	2.3	5.071E-03	2.95E-03	0.486
(10,10)	2.025	2.95	1.735	2.16	1.932E-03	9.53E-04	0.493
(15,15)	2.351	3.396	2.09	3.018	9.97E-04	5.57E-04	0.559
(20,20)	2.354	2.751	2.139	2.501	5.98E-04	3.65E-04	0.61
(5,4)	1.823	3.345	1.457	3.158	5.54E-03	2.22E-03	0.4
(10,8)	2.234	2.99	1.914	3.011	2.302E-03	1.10E-03	0.48
(10,9)	2.701	3.0247	1.955	3.089	2.10E-03	1.02E-03	0.487
(20,18)	2.123	2.798	2.455	2.533	5.61E-04	3.97E-04	0.6
(20,19)	2.373	3.1	2.165	2.818	2.59E-04	1.58E-04	0.608

Table (6) $u_1 = 0.8, u_2 = 0.8, \theta_1 = 1, \theta_2 = 2$ *and* $R = 0.333$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	0.991	1.83	0.799	1.475	5.2911E-04	5.40E-04	1.017
(10,10)	1.471	2.325	1.268	2.405	3.37E-04	3.4E-04	1.011
(15,15)	0.132	1.923	1.222	1.717	2.1E-04	2.2E-04	1.004
(20,20)	0.999	2.36	1.011	2.153	1.41E-04	1.4E-04	1.001
(5,4)	0.923	1.635	0.744	1.717	7.12E-04	5.9E-04	0.978
(10,8)	1.775	2.669	1.53	2.253	3.934E-04	3.8E-04	0.971
(10,9)	1.7	1.929	1.245	1.647	3.6E-04	3.6E-04	0.992
(20,18)	1.333	2.454	1.111	2.222	1.5E-04	1.5E-04	0.985
(20,19)	1.532	2.478	1.212	2.323	1.5E-04	1.4E-04	0.993

Table (7) $u_1 = 1, u_2 = 1, \theta_1 = 1, \theta_2 = 2$ *and* $R = 0.333$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	0.725	1.784	0.579	1.524	1.54E-03	1.60E-03	1.045
(10,10)	0.888	1.967	0.657	1.685	8.20E-04	8.48E-04	1.034
(15,15)	0.965	2.452	0.875	2.179	5.03E-04	5.15E-04	1.024
(20,20)	1.121	2.314	0.924	2.014	3.38E-04	3.44E-04	1.017
(5,4)	1.149	2.131	0.899	1.704	1.720E-03	1.729E-03	1.005
(10,8)	1.1545	2.553	0.983	2.141	9.589E-04	9.527E-04	0.993
(10,9)	1.254	2.178	0.999	1.866	8.842E-04	8.974E-04	1.015
(20,18)	1.222	2.236	1.021	2.017	3.701E-04	3.705E-04	1.088
(20,19)	1.235	2.353	1.0222	2.131	3.533E-04	3.567E-04	1.01

Table (8) $u_1 = -1, u_2 = -1, \theta_1 = 1, \theta_2 = 2$ *and* $R = 0.333$

(n,m)	$MSE1$	$MSE2$	$LB1$	$LB2$	R_{LB}	R_{MSE}	$RE3$
(5,5)	1.533	1.841	1.332	1.6	1.55E-03	1.3E-03	0.893
(10,10)	1.232	2.031	1.222	1.846	8.51E-04	7.6E-04	0.896
(15,15)	1.325	2.194	1.25	2.041	5.23E-04	4.7E-04	0.909
(20,20)	1.424	2.015	1.24477	2.145	3.50E-04	3.2E-04	0.919
(5,4)	1.254	2.159	1.33	1.876	1.744E-03	1.5E-03	0.864
(10,8)	1.111	2.162	1.145	1.937	1.002E-03	8.6E-04	0.867
(10,9)	1.675	1.831	1.522	1.654	9.205E-04	8.1E-04	0.883
(20,18)	1.254	2.428	1.425	2.279	3.845E-04	3.4E-04	0.908
(20,19)	1.92	2.921	1.811	2.749	3.667E-04	3.3E-04	0.914

We can study the different between the cases of the risk functions in the different case of sample sizes equal sample sizes and not equals ,in different times; it will be clear from figures mentioned below:

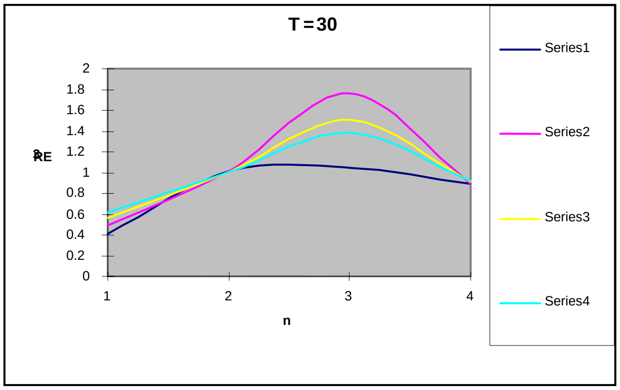

Figure (1): to Compare between the proposed estimators of $RE3$ in different cases of the sample sizes $n, m = 5,10,15$ and 20 and for different values of u_1, u_2 in case of time =30 hours.

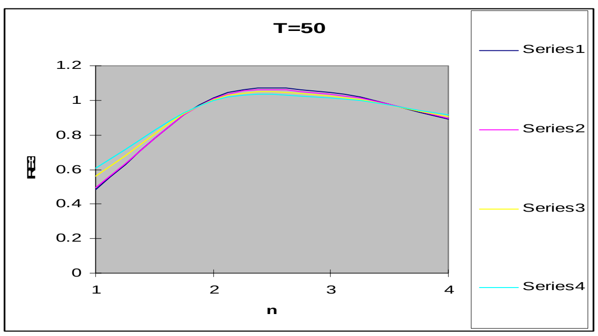

Figure (2): to Compare between the proposed estimators of $RE3$ in different cases of the sample sizes $n, m = 5,10,15$ and 20 and for different values of u_1, u_2 in case of time =50 hours.

5. Illustrative Example

The simulation study shows that $R_L(\hat{\theta}_{LB1}, \hat{\theta}_{LB2})$ is preferable than $R_L(\hat{\theta}_{MSB1}, \hat{\theta}_{MSB2})$. This section will provide a true example on the life test from Weibull distribution with type-I censored data for failure terminated data given in blow tables come. These $T_1, \ldots, T_{10}$ failure times corresponding to devices $x_1, \ldots, x_{10}$,and $U_1, \ldots, U_{10}$ failure times corresponding to devices $y_1, \ldots, y_{10}$

Table 9. Numbers.

77.668	16.263	116.729	399.071
118.077	182.737	102.947	6.17
87.592	12.411	49.022	72.435
6.171	7.400	12.411	21.49
89.601	272.005	199.458	50.668

Using the statistics of total time. we obtain the following point estimator of the mean life:

$$\text{Mean life} = \frac{\text{Total test time}}{\text{sample numeber}} \quad (41)$$

Then for x and y the mean life will be:

$$\text{Meanlife}_x = \frac{\sum T}{n} = \frac{859.925}{10} = 85.992 \quad (42)$$

and

$$\text{Meanlife}_y = \frac{\sum U}{m} = \frac{1030.402}{10} = 103.04. \quad (43)$$

The reciprocal of mean life, yields the point estimate of the device failure rate for x and y :

$$\text{failure rate} = \frac{1}{\text{mean life}}. \quad (44)$$

Then the failure rate for x and y will be:

$$\text{failurerate}_x = \frac{1}{85.9925} = 0.0116. \quad (45)$$

and

$$\text{failurerate}_y = \frac{1}{103.0402} = 0.0097. \quad (46)$$

We then use the appropriate statistical reliability to calculate the CI in the Weibull case .The formula to obtain a Weibull CI for true , but unknown mean for x and y indexed by $\hat{\theta}_1$ and $\hat{\theta}_2$ respectively with a confidence level $100\ (1-\alpha)\%$,is given by.

$$[F_{1-\alpha/2}(2n,2m)(\frac{\Sigma T}{\Sigma U} + F_{1-\alpha/2}(2n,2m))^{-1}, F_{\alpha/2}(2n,2m)(\frac{\Sigma T}{\Sigma U} + F_{\alpha/2}(2n,2m))^{-1}]. \quad (47)$$

From the last equation we find the CL is 0.9999 and we use it to obtain 90%confidence bound for the Weibull mean, the 90% upper bound for unknown mean:

$$\hat{\theta}_1 = 2.195 \text{ and } \hat{\theta}_2 = 2.195 \quad (48)$$

Then the failure rate is

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2} = 0.499 \quad (49)$$

Now we find the failure rate for mean square error loss function case and LINEX loss function case:

I-for mean square error loss function:

Before finding the value of failure rate $R_L(\hat{\theta}_{MSB\,1}, \hat{\theta}_{MSB\,2})$,we should find the value of $\hat{\theta}_{MSB\,1}$ and $\hat{\theta}_{MSB\,2}$ from equations (17) and (18) ,so:

$$\hat{\theta}_{MSB\,1} = 0.304 \text{ and } \hat{\theta}_{MSB\,2} = 0.0158 \quad (50)$$

Where $a,b,c,d > 0, u_1, u_2 \neq 0$, and for S_1, S_2 calculated by MATHCAD program and we get their values as the following

$S_1 = -2.604$, $S_2 = -2.794$. (51)

Now from equation (39) we can find the value of

$R_L(\hat{\theta}_{MSB\,1}, \hat{\theta}_{MSB\,2}) = 0.9505$ (52)

II-for LINEX loss function:

Before finding the value of failure rate $R_L(\hat{\theta}_{LB\,1}, \hat{\theta}_{LB\,2})$,we should find the value of $\hat{\theta}_{LB\,1}$ and $\hat{\theta}_{LB\,2}$ from equations (31) and (32) ,so:

$\hat{\theta}_{LB\,1} = 0.0235$ and $\hat{\theta}_{LB\,2} = 0.01244$. (53)

By using equation (35) we can find the value of

$R_L(\hat{\theta}_{LB\,1}, \hat{\theta}_{LB\,2}) = 0.653$ (54)

Using equation (40) to find failure rate for mean square error loss function case and LINEX loss function case:

$$RE = \frac{0.9505}{0.653} = 1.4555 \quad (55)$$

RE is greater than one which indicates that the proposed estimators $R_L(\hat{\theta}_{LB\,1}, \hat{\theta}_{LB\,2})$ are preferable to $R_L(\hat{\theta}_{MSB\,1}, \hat{\theta}_{MSB\,2})$.

4. Conclusion

In this paper, the Bayes estimation of stress-strength parameter for two Weibull distributions under type-I censoring has been considered. The LINEX loss function and square error loss function have been proposed. Extensive simulations are performed to check the performance of the two estimators, and it's observed that the LINEX loss function is preferable to the square errors loss function. Also, for the illustrative example which demonstrated agreed with the paper results.

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