Stresses in the sphere rests on a rigid plane horizontal surface

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Abstract: In this paper, Analytical stresses are obtained for the sphere rests on a rigid plane horizontal surface. It is assumed that the support reaction consists of a concentrated vertical force equal to the weight of the sphere. We are started with the solution for a point force acting on the surface of a half-space and determine the tractions on an imaginary spherical surface passing through the point of application of the force, then complete the solution by superposing appropriate spherical harmonics. The results differ significantly from the classical elasticity solutions that are based on the assumption that the body is fully formed before the loading is applied. The self-equilibrated tractions due to self-weight and the concentrated force alone and with the approximations obtained using $n = 2$ and $n = 4$. The process is clearly convergent and as with Fourier series approximations, the error exhibits more zero crossings as the number of terms increases.

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1. Introduction

The majority of contacts fall into one of two classes; those where at least one of the contacting bodies is convex, so that the contact size depends on the normal load – an incomplete contact, and those where the contact-defining body has a surface profile with distinct discontinuities in surface slope which define the contact size – a complete contact. Partial slip contact problems are of great practical interest because the damage caused by slip within notionally stationary contacts encourages the nucleation of fatigue cracks. Most solutions to partial slip problems within the literature are based on half-plane theory, and employ the usual Amontons–Coulomb frictional law, which remains the most appropriate model for most metallic contacts, and is also used here. The pioneering solution describes the response of a Hertzian contact subject to a constant normal force and oscillatory shear (Cattaneo, 1938). Mindlin revisited the problem ten years later, and extended the solution to consider more complex sequences of loading (Mindlin, 1949; Mindlin & Dereciewicz, 1953). The most important development in this family of solutions since then has been the simultaneous discovery by Jager (1998) and by Ciavarella (1998) that the 'corrective' shear traction distribution within the stick zone is similar in form to the contact pressure sustained by the contact under a lighter load-this is true for any half-plane contact. As half-plane theory can model only incomplete contacts the contact pressure always falls smoothly to zero at the contact edge, so that there is always a region of slip present under a monotonically increasing load, and which starts at the edge. Recently, the authors examined the case of two similar elastic cylinders pressed end-on-end and twisted, and investigated the development of slip. That problem incorporates an unusual type of contact-where the size is simultaneously defined by both bodies-and where the shearing traction is anti-plane with respect to a diametral plane (Kartal, Hills, Nowell, & Barber, 2010).

2. Preliminaries

2.1. The Boussinesq solution

We shall now apply similar arguments to solve the Boussinesq problem, in which a point force F in the $z-direction$ is applied at the origin $R = 0$ on the surface $z = 0$ of the half-space $z \succ 0$.

We note that the force is normal to the surface, so that there is no tangential traction at any point on the surface — i.e.

$$\sigma_{zx} = \sigma_{zy} = 0 \ ; \ all\ x, y,\ z = 0 \qquad (1)$$

We therefore seek a suitable partial integral of $1/R$ to be dimensionless in R and singular at the origin, but otherwise to be continuous and harmonic in $z \succ 0$.

It is easily verified that the function

$$\varphi = \int_{-\infty}^{0} \frac{d\varsigma}{\sqrt{x^2 + y^2 + (z-\varsigma)^2}} = \ln(R + z) \qquad (2)$$

The force applied at the origin is

$$F = -2\pi\int_0^\infty r\sigma_{zz}(r,h)dr = -6\pi h^3\int_0^\infty \frac{rdr}{(r^2+h^2)^{5/2}} = -2\pi \tag{3}$$

and hence the stress field due to a force F in the $z-direction$ applied at the origin is obtained from the potential

$$\varphi = -\frac{F}{2\pi}\ln(R+z) \tag{4}$$

2.2 Other singular solutions

We have already shown how the singular solution in $(R+z)$ can be obtained from $1/R$ by partial integration, which of course is a form of superposition. A whole sequence of axially symmetric solutions can be obtained in the same way. Defining

$$\phi_0 = \frac{1}{R} \tag{5}$$

We obtain

$$\begin{aligned}
&\phi_{-1} = \ln(R+z)\ ;\ \phi_{-2} = z\ \ln(R+z) - R \\
&\phi_{-3} = \frac{1}{4}\{(2z^2 - r^2)\ln(R+z) - 3Rz + r^2\} \\
&\phi_{-4} = \frac{1}{36}\{3(2z^3 - 3zr^2)\ln(R+z) + 9zr^2 \\
&\quad -11z^2R + 4r^2R\}\ ;
\end{aligned} \tag{6}$$

By the operation

$$\phi_{n-1}(r,z) = \int_{-\infty}^{0}\phi_n(r,(z-\varsigma))d\varsigma\ . \tag{7}$$

The inverse operation is one of differentiation, so that

$$\phi_n = \frac{\partial\phi_{n-1}}{\partial z}\ . \tag{8}$$

We can therefore also extend the sequence to functions with stronger singularities such as

$$\phi_1 = -\frac{z}{R^3};\phi_2 = \frac{3z^2}{R^5} - \frac{1}{R^3};\phi_3 = -\frac{15z^3}{R^7} + \frac{9z}{R^5}. \tag{9}$$

If the half-space is indented by a frictionless punch, so that the surface $z=0$ is subjected to normal tractions only, a simple formulation can be obtained by combining solution and defining a relationship between ϕ and ω in order to satisfy identically the condition $\sigma_{zx} = \sigma_{zy} = 0$ on $z=0$.

We write

$$\phi = (1-2\nu)\varphi\ ;\ \omega = \frac{\partial\varphi}{\partial z} \tag{10}$$

Obtaining

$$\sigma_{zx} = z\,\frac{\partial^3\varphi}{\partial x\,\partial z^2}\ ;\ \sigma_{zy} = z\,\frac{\partial^3\varphi}{\partial y\,\partial z^2}\ , \tag{11}$$

Therefore

$$\sigma_{RR} = \frac{\partial^2\phi}{\partial R^2} + R\cos\beta\frac{\partial^2\omega}{\partial R^2} - 2(1-\nu)\frac{\partial\omega}{\partial R}\cos\beta \tag{12}$$

$$\begin{aligned}
\sigma_{\theta\theta} = &\frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial\phi}{\partial\beta} + \frac{1}{R^2\sin^2\beta}\frac{\partial^2\phi}{\partial\theta^2} \\
&-(1-2\nu)\frac{\partial\omega}{\partial R}\cos\beta + \frac{2\nu}{R}\frac{\partial\omega}{\partial\beta}\sin\beta \\
&+\frac{\cos^2\beta}{R\sin\beta}\frac{\partial\omega}{\partial\beta} + \frac{\cot\beta}{R\sin\beta}\frac{\partial^2\omega}{\partial\theta^2}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\sigma_{\beta\beta} = &\frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\phi}{\partial\beta^2} + (1-2\nu)\frac{\partial\omega}{\partial R}\cos\beta \\
&+\frac{2(1-\nu)}{R}\frac{\partial\omega}{\partial\beta}\sin\beta + \frac{\cos\beta}{R}\frac{\partial^2\omega}{\partial\beta^2}
\end{aligned} \tag{14}$$

$$\begin{aligned}
\sigma_{\beta R} = &\frac{1}{R}\frac{\partial^2\phi}{\partial\beta\partial R} - \frac{1}{R^2}\frac{\partial\phi}{\partial\beta} + (1-2\nu)\frac{\partial\omega}{\partial R}\sin\beta \\
&-\frac{2(1-\nu)}{R}\frac{\partial\omega}{\partial\beta}co\,s\,\beta + \cos\beta\frac{\partial^2\omega}{\partial\beta\partial R}
\end{aligned} \tag{15}$$

$$\sigma_{\theta R} = \sigma_{\theta\beta} = 0 \tag{16}$$

3. Stress field due to rests on a rigid plane

3.1. Governing equations

The general solution for a solid sphere with prescribed surface tractions can be obtained using the spherical harmonics. The addition of the singular harmonics permits a general solution to the axisymmetric problem of the hollow sphere, but the corresponding non-axisymmetric. Using equations (4)-(10), we obtain

$$\varphi_0 = (1-2\nu)\varphi = -\frac{F(1-2\nu)}{2\pi}\ln(R+z) \tag{17}$$

$$\omega_0 = \frac{\partial\varphi}{\partial z} = -\frac{F}{2\pi R} \tag{18}$$

These can be written in spherical polar coordinates centred on the point of application of the force as

$$\varphi_0 = -\frac{F(1-2\nu)}{2\pi}\ln(R+R\cos\beta) \tag{19}$$

$$\omega_0 = -\frac{F}{2\pi R} \tag{20}$$

and the corresponding non-zero stress components are obtained by substitution into equations (12)-(16) as

$$\sigma_{RR} = \frac{F[1-2\nu-2(2-\nu)\cos\beta]}{2\pi R^2} \tag{21}$$

$$\sigma_{\theta\theta} = -\frac{F(1-2\nu)(1-\cos\beta-\cos^2\beta)}{2\pi R^2(1+\cos\beta)} \quad (22)$$

$$\sigma_{\beta\beta} = -\frac{F(1-2\nu)\cos^2\beta}{2\pi R^2(1+\cos\beta)} \quad (23)$$

$$\sigma_{\beta R} = -\frac{F(1-2\nu)\sin\beta\cos\beta}{2\pi R^2(1+\cos\beta)} \quad (24)$$

$$\sigma_{\theta R} = \sigma_{\theta\beta} = 0 \quad (25)$$

Points on the surface of the sphere are defined by the equation

$$R = 2\alpha\cos\beta \quad (26)$$

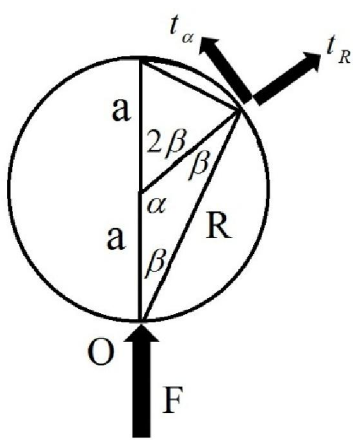

Figure 1. Configuration, loading and geometry.

As shown in figure 1. At this point we have

$$\sigma_{RR} = \frac{F[1-2\nu-2(2-\nu)\cos\beta]}{8\pi a^2\cos^2\beta} \quad (27)$$

$$\sigma_{\theta\theta} = -\frac{F(1-2\nu)(1-\cos\beta-\cos^2\beta)}{8\pi a^2\cos^2\beta(1+\cos\beta)} \quad (28)$$

$$\sigma_{\beta\beta} = -\frac{F(1-2\nu)}{8\pi a^2(1+\cos\beta)} \quad (29)$$

$$\sigma_{\beta R} = -\frac{F(1-2\nu)\sin\beta}{8\pi a^2\cos\beta(1+\cos\beta)} \quad (30)$$

$$\sigma_{\theta R} = \sigma_{\theta\beta} = 0 \quad (31)$$

However, to find the implied tractions on the spherical surface, we need to rotate the local coordinate system clockwise through an angle β, obtaining the radial and tangential tractions

$$t_R = \sigma_{RR}\cos^2\beta + \sigma_{\beta\beta}\sin^2\beta + 2(-\sigma_{\beta R})(-\sin\beta)\cos\beta \quad (32)$$

$$t_\alpha = (\sigma_{\beta\beta}-\sigma_{RR})(-\sin\beta)\cos\beta + (-\sigma_{\beta R})(\cos^2\beta-\sin^2\beta) \quad (33)$$

In these equations, notice that the sign convention for a Cartesian coordinate system (x, y) aligned with (R, β) at the surface would involve a negative shear stress ($\sigma_{xy} = -\sigma_{\beta R}$) and the clockwise rotation introduces a negative sign into the terms involving $\sin\beta$. The direction of the shear traction t_α is chosen so as to be consistent with the angle α subtended at the centre of the sphere, defined such that $\alpha = 0$ corresponds to the point of application of the force. From figure 1, we then have

$$2\beta = \pi - \alpha \quad (34)$$

Substituting from (27)-(31) into (32), (33) and simplifying, we obtain

$$t_R = \frac{F[4(1-2\nu)-(7-8\nu)\cos\beta]}{8\pi a^2} \quad (35)$$

$$t_\alpha = \frac{F[2(1-2\nu)-3\cos\beta-(7-8\nu)\cos^2\beta]\sin\beta}{8\pi a^2(1+\cos\beta)\cos\beta} \quad (36)$$

Or in terms of α,

$$t_R = \frac{F[4(1-2\nu)-(7-8\nu)\sin(\alpha/2)]}{8\pi a^2} \quad (37)$$

$$t_\alpha = -\frac{F[3+6\sin(\alpha/2)-(7-8\nu)\cos\alpha]\cos(\alpha/2)}{16\pi a^2(1+\sin(\alpha/2))\sin(\alpha/2)} \quad (38)$$

These expression are not of Fourier form in the angle α and hence the process of adding terms to satisfy the traction boundary conditions on the sphere will be more complex than in the two-dimension case. More seriously, the traction t_α is singular as $\alpha \to 0$ — i.e. at the point of application of the force. This result was first remarked by Sternberg and Rosenthal. It might still be possible to satisfy the boundary condition using an infinite series of spherical harmonics, but the series would probably be only very slowly convergent because of the singularity.

To prevent this problem and improve the convergence of the series, we need to superpose additional potentials chosen so as to cancel this singularity. As a preliminary to this process, we can expand (38) near $\alpha = 0$ by writing

$$\sin\alpha \approx \alpha; \sin(\alpha/2) \approx \frac{\alpha}{2}; \quad \cos\alpha \approx 1-\frac{\alpha^2}{2}; \cos(\alpha/2) \approx \frac{1-\alpha^2}{8}, \quad (39)$$

Obtaining

$$t_\alpha \approx \frac{F(1-2\nu)}{2\pi a^2\alpha} - \frac{F(5-4\nu)}{8\pi a^2} + O(\alpha) \quad (40)$$

As $\alpha \to 0$.

To choose a suitable potential to cancel the first term, notice that the surface is very nearly plane when $\alpha \to 0$, so we can look for potentials giving shear tractions on the surface of the half plane with this singular form. This may seem curious, since the original Boussinesq solution gave identically zero tractions on this plane, but we note that the singular tractions are one order lower than the singularity associated with the point force, which is order R^{-2}. We therefore choose additional potentials from equation (6) that are one order less singular than those in (19), (20). Notice that these will introduce singular tractions both t_R and t_α, the former is undesirable, so we use the same combination make the dominant singular term in t_R be zero.

We therefore choose

$$\phi_1 = 2(1-\nu)CR[\cos\beta\ln(R+R\cos\beta)-1] \quad (41)$$

$$\omega_1 = C\ln(R+R\cos\beta) \quad (42)$$

We find the tractions due to this potential as $\alpha \to 0$ to be

$$t_R \approx -\frac{C}{2a} + O(\alpha) \quad (43)$$

$$t_\alpha \approx -\frac{C}{a\alpha} + \frac{C(1-2\nu)}{2a} + O(\alpha) \quad (44)$$

Superposing this on the original stress function, it is clear that we can cancel the unwanted singularity by choosing

$$\frac{F(1-2\nu)}{2\pi a^2} - \frac{C}{a} = 0\ , \quad (45)$$

Or

$$C = \frac{F(1-2\nu)}{2\pi a} \quad (46)$$

With this choice, the tractions everywhere on the surface of the sphere are bounded, but we notice from equation (44) that the traction t_α will be non-zero at $\alpha = 0$. Now it is easily verified that all the axisymmetric spherical harmonics give zero values of shear traction $\sigma_{R\beta}$ on the axis $\beta = 0$. In other words, although the magnitude of the traction is continuous, it's direction changes discontinuously at the origin. This is itself a kind of singularity.

This additional singularity can also be removed by superposing the next higher order potentials, once again chosen form (6) so as to satisfy – i.e.

$$\phi_2 = \frac{(1-\nu)AR^2}{2}[(2\cos^2\beta - \sin^2\beta) \times\cos\beta\ln(R+R\cos\beta) - 3\cos\beta + \sin^2\beta] \quad (47)$$

$$\omega_2 = AR[\cos\beta\ln(R+R\cos\beta)-1] \quad (48)$$

We find the traction due to the superposition of all the above potentials at $\alpha \to 0$ to be

$$t_R \approx -\frac{F(1-2\nu)}{2\pi a^2} + O(\alpha) \quad (49)$$

$$t_\alpha \approx -\frac{F(1+12\nu-16\nu^2)}{8\pi a^2} + A + O(\alpha). \quad (50)$$

And hence we can eliminate the cowlick by choosing

$$A = \frac{F(1+12\nu-16\nu^2)}{8\pi a^2}. \quad (51)$$

Thus, a suitably smooth form of the point force solution for the sphere is provided by the potentials

$$\begin{aligned}\phi &= \phi_0 + \phi_1 + \phi_2 \\ &= -\frac{F(1-2\nu)}{2\pi}\ln(R+R\cos\beta) \\ &+ \frac{2F(1-2\nu)(1-\nu)R[\cos\beta\ln(R+R\cos\beta)-1]}{2\pi a} \\ &+ \frac{F(1+12\nu-16\nu^2)(1-\nu)R^2}{16\pi a^2} \\ &\times[(2\cos^2\beta-\sin^2\beta)\cos\beta\ln(R+R\cos\beta) \\ &-3\cos\beta+\sin^2\beta]\end{aligned} \quad (52)$$

$$\begin{aligned}\omega &= \omega_0 + \omega_1 + \omega_2 \\ &= -\frac{F}{2\pi R} + \frac{F(1-2\nu)\ln(R+R\cos\beta)}{2\pi a} \\ &+ \frac{F(1+12\nu-16\nu^2)R}{8\pi a^2} \\ &\times[\cos\beta\ln(R+R\cos\beta)-1]\end{aligned} \quad (53)$$

To confirm that the tractions now remaining to be removed are smooth, we plot them in figure 2 as functions of α.

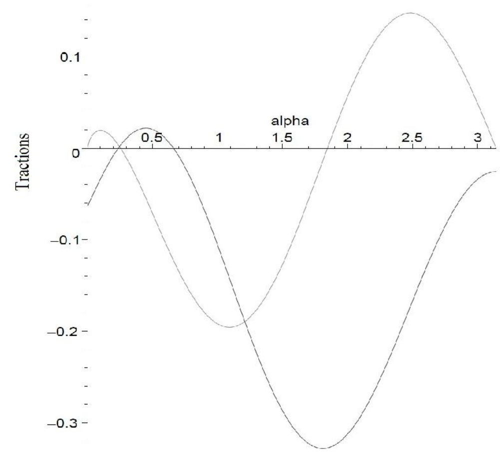

Figure 2. Tractions on the spherical surface associated with the stress functions of equation (52, 53). The curve passing through the origin is the shear traction t_α.

3.2. Gravitational

The body force due to self-weight is conveniently introduced as a hydrostatic stress

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\alpha\alpha} = -\rho g\tilde{R}\cos\alpha\ , \quad (54)$$

Where $\tilde{R}$ is here measured from the centre of the sphere. This adds an additional term $-\rho ga\cos\alpha$. Into the traction component t_R, whilst leaving t_α unchanged. Also, we note that the force F must support the weight of the sphere, so

$$F = \frac{4\pi\rho ga^3}{3} . \tag{55}$$

3.3 Spherical harmonics

To complete the solution, we superpose a series of spherical harmonics. Thus, we add the new potentials

$$\phi_3 = \sum_{i=1}^{n} A_i \tilde{R}^{i+1} P_{i+1}(\cos\alpha)\omega = \sum_{i=1}^{n} B_i \tilde{R}^i P_i(\cos\alpha), \tag{56}$$

evaluate the additional tractions on the surface $R = a$. since we can only use a finite number of terms in the series, we can satisfy the traction-free boundary condition might be chosen, but the must convergent is to use a Galerkin or 'weighted residual' method. For example, if we write the approximate tractions in the form

$$\tilde{t}_R = t_R^P + \sum_{i=1}^{n} A_i t_{Ri}^A(\alpha) + \sum_{i=1}^{n} B_i t_{Ri}^B(\alpha) \tag{57}$$

$$\tilde{t}_\alpha = t_\alpha^P + \sum_{i=1}^{n} A_i t_{\alpha i}^A(\alpha) + \sum_{i=1}^{n} B_i t_{\alpha i}^B(\alpha) , \tag{58}$$

we can define an error measure

$$E = \int_0^\alpha (\tilde{t}_R^2 + \tilde{t}_\alpha^2) d\alpha . \tag{59}$$

An optimal choice of the constants A_i, B_i can then be made by requiring

$$\frac{\partial E}{\partial A_i} = 0 \ ; \ \frac{\partial E}{\partial B_i} = 0 , \tag{60}$$

for $i = 1, n$. Notice that the error measure has been weighted uniformly in $0 \prec \alpha \prec \pi$. An alternative choice here would be to weight according to the volume of surface of the sphere, which would introduce a factor of $\sin\alpha$ into the integral. This would probably give better accuracy near the equator (where there is more surface area) and less near the poles.

The effect of this process is to weight the traction-free condition according to the practicable in mathematica. In Figure 3 we present the self-equilibrated tractions t_R due to self weight and the concentrated force alone and with the approximations obtained using $n = 2$ and $n = 4$ respectively. The process is clearly convergent and as with Fourier series approximations, the error exhibits more zero crossings as the number of terms increases. Oscillations near $\alpha = 0$ (Gibb's phenomenon) are to be expected with large numbers of terms, but this effect has been to some extent mitigated by the removal of the stronger discontinuous effects in the above analysis.

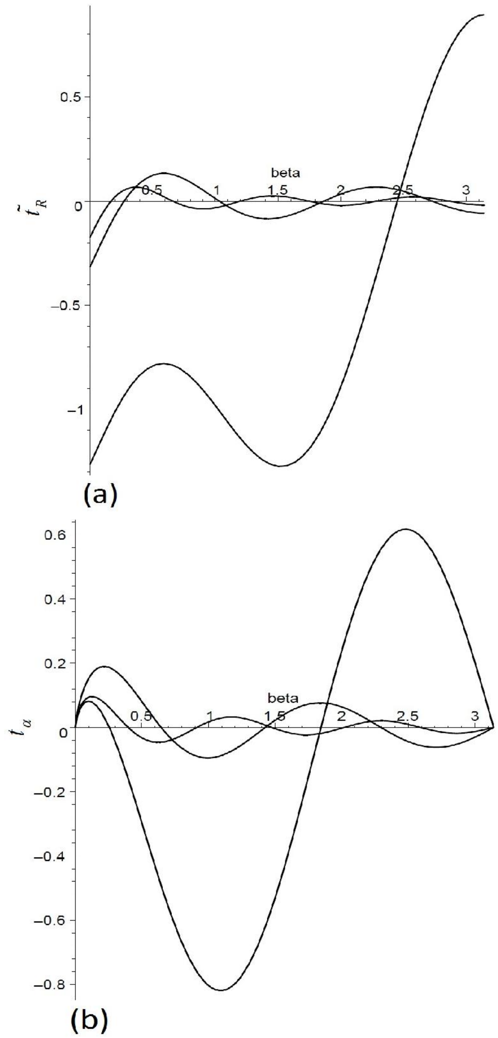

Figure 3. Tractions in the approximate solution (a) $\tilde{t}_R$ and (b) $\tilde{t}_\alpha$, normalized by ρga for the case $\nu = 0.3$.

4. Conclusion

To obtain the Stresses in the sphere rests on a rigid plane horizontal surface, equations are solved utilizing the Boussinesq solution method and determined the tractions on an imaginary spherical surface passing through the point of application of the force, then complete the solution by superposing appropriate spherical harmonics.

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